TURK'S HEAD KNOTS WITH AT LEAST THREE LEADS AND THREE BIGHTS ARE NOT PERFECTLY STRAIGHT

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ABSTRACT. Owad defined the straight number $\mathrm{str}(K)$ of a knot, the least number of crossings of a diagram in which one straight strand meets all crossings, and called K perfectly straight if $\mathrm{str}(K)$ equals the crossing number. Di Prisa and Şavk asked whether the Turk's head knot $Th(p,q)$ is perfectly straight when $p \geq 3$ and $q < p$ (Problem A of their survey). We prove that $Th(p,q)$ is not perfectly straight for all $p, q \geq 3$ with $\gcd(p,q) = 1$. For $q < p$ this answers Problem A when $q \geq 3$; for $q > p$ it is a theorem of Owad, of which we give an independent proof. The proof combines an exact count on the standard diagram (its longest repetition-free run of crossings has length $2q + p - 2\lfloor q/p \rfloor - 3$, less than the crossing number $(p-1)q$) with the uniqueness of the shadow of the minimal diagram up to homeomorphism of the sphere, which follows from the flyping theorem. For $q = 2$ we show, using the two-bridge lemma of a separate note of ours [Q2] and a theorem of Belousov and Malyutin, that $Th(p,2)$ is perfectly straight for every odd $p \geq 3$. Using the two-bridge lemma of [Q2], this answers Problem A for $2 \leq q < p$: for $p \geq 3$ and $2 \leq q < p$, $Th(p,q)$ is perfectly straight if and only if $q = 2$.

1. INTRODUCTION

The Turk's head link $Th(p,q)$ is the closure of the p-strand braid β^q, $\beta = \sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\cdots\sigma_{p-1}^{\pm 1}$. It is a knot if and only if $\gcd(p,q) = 1$ [DS, §2 and §4], and its crossing number is $c(Th(p,q)) = (p-1)q$ [DS, §4.2]. Owad's weaving knot $W(n,m)$ [Ow2, Definition 3.1 and §3] is $Th(n,m)$ up to mirror image.

Following Owad [Ow1, Ow2], a diagram is in *straight position* if it has a single straight strand that contains all of the crossings. The *straight number* $\mathrm{str}(K)$ of a knot K is the minimum number of crossings over all diagrams of K in straight position [Ow2, Definition 2.1], and K is *perfectly straight*

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if $\mathrm{str}(K) = c(K)$. Owad proved [Ow2, Theorem 3.3] that $W(n,m)$ is not perfectly straight for $n \geq 3$, $m \geq n+1$, $\gcd(n,m) = 1$. Di Prisa and Şavk [DS, §6.1] posed:

> **Problem A.** *Determine whether the Turk's head knot $Th(p,q)$ is perfectly straight when $p \geq 3$ and $q < p$.*

Theorem 1. *Let $p, q \geq 3$ with $\gcd(p,q) = 1$, and write $b = \lfloor q/p \rfloor$. Every diagram of $Th(p,q)$ with $(p-1)q$ crossings has a longest repetition-free run of crossings of length exactly $2q+p-2b-3 < (p-1)q$. In particular $Th(p,q)$ is not perfectly straight:* $\mathrm{str}(Th(p,q)) > (p-1)q$.

For $q < p$ (so $b = 0$ and the length is $2q+p-3$) this answers Problem A for $q \geq 3$. For $q > p$ it is Owad's Theorem 3.3; our proof is independent of his: the count is Proposition 9, the uniqueness of the minimal diagram is Proposition 11. Remark 10 compares the two arguments. Since str is defined for knots, links ($\gcd(p,q) > 1$) are not considered; for $p \geq 3$ the case $q = p$ gives a link. We set the case $q = 1$ of Problem A aside: $Th(p,1)$ is the closure of β, in which $\sigma_{p-1}^{\pm 1}$ occurs once, so Markov destabilisations reduce it step by step to the trivial 1-strand braid; hence $Th(p,1)$ is the unknot (as in [DS, Theorem A]). Like the survey from its §4 on, we therefore assume $q \geq 2$; the straight number of the unknot is not discussed here.

Theorem 2. *Using the two-bridge lemma of* [Q2, Lemma 2]*: for every odd $p \geq 3$ the knot $Th(p,2)$ is perfectly straight.*

The lemma of [Q2] states that $Th(p,2)$ is a two-bridge knot; it is cited, not reproved here, and Theorem 2 and Corollary 3 depend on it.

Corollary 3. *Using the two-bridge lemma of* [Q2, Lemma 2]*: for $p \geq 3$, $2 \leq q < p$, $\gcd(p,q) = 1$, the knot $Th(p,q)$ is perfectly straight if and only if $q = 2$.*

The standard diagram of $Th(5,2) = 8_{12}$ [DS, Table 1] is not in straight position (Remark 13), so for $q = 2$ a straight minimal diagram is in general a different diagram.

2. Straightness as a property of the Gauss sequence

Diagrams are taken to be PL; every diagram is planar isotopic to a PL one. For a knot diagram D with $n \geq 1$ crossings, let $\gamma(D)$ be the cyclic sequence of length $2n$ of crossings met when one walks once around the knot. A *run* is a cyclically consecutive block of entries of $\gamma(D)$; it is *repetition-free* if its entries are pairwise distinct. Let $\mathrm{maxrun}(D)$ be the maximal length of a repetition-free run; $\mathrm{maxrun}(D) \leq n$. A diagram is *composed of two simple arcs* (a *semi-meander diagram* [BM, (C1)]) if the knot is the union of two arcs with common endpoints, neither of which crosses itself.

Lemma 4. *(a) For a diagram D in $\mathbb{R}^2$ with n crossings the following are equivalent: (i) D is in straight position after a planar isotopy; (ii) D is composed of two simple arcs; (iii)* $\mathrm{maxrun}(D) = n$.

(b) *K is perfectly straight if and only if some diagram of K with $c(K)$ crossings is in straight position.*
(c) $\mathrm{maxrun}(D)$ *depends only on the shadow of D, up to homeomorphisms of S^2 of either orientation. In particular it is unchanged by moving the point at infinity, by reflection, by crossing changes, and by reversing the orientation.*

Proof. (a) (i)⇒(iii): a planar isotopy does not change γ. Every crossing lies on the straight strand, and a straight segment passes through each point of the plane at most once, so the strand passes each crossing exactly once, and these n passages are consecutive in $\gamma(D)$. (iii)⇒(ii): let A be an arc of the knot that starts just before the first and ends just after the last passage of a repetition-free run of length n. It passes n distinct crossings once each, so it does not cross itself. Each crossing is passed twice by the knot and once by A, so the complementary arc B passes each crossing once and does not cross itself either. (ii)⇒(i): if $D = A \cup B$ with A, B simple, every crossing is a point of $A \cap B$, so the arc A contains all crossings. The arc A is a PL arc; extended to a PL simple closed curve, it is carried onto a straight segment by a PL homeomorphism of $\mathbb{R}^2$ (PL Schoenflies theorem), which may be taken orientation-preserving, and such a homeomorphism is isotopic to the identity through PL homeomorphisms, so the image is again a PL diagram. So a planar isotopy carries A onto a straight segment, and the image of D is in straight position. This is the step by which Owad passes from a diagram made of two sub-arcs without self-crossings to straight position [Ow2, §1]. (b) A diagram in straight position is a diagram, so $\mathrm{str}(K) \geq c(K)$; equality holds exactly when a straight diagram with $c(K)$ crossings exists. (c) $\gamma(D)$ is the vertex sequence of the straight-ahead walk on the 4-valent shadow: one leaves each vertex by the half-edge opposite to the one of arrival. Every homeomorphism of S^2 preserves or reverses the cyclic order at a vertex, hence preserves opposition. Crossing data do not enter. Reversal reverses γ. □

3. The standard diagram

Let Π be the closure of β^q. Write (k, i), $k \in \mathbb{Z}/q$, $1 \leq i \leq p-1$, for the letter $\sigma_i^{\pm 1}$ in the k-th copy of β (the k-th *block*); it exchanges positions i and $i+1$. Put $L = p-1$ and $n = qL$.

Lemma 5. *A strand entering block k at position 1 meets $(k,1), \ldots, (k,p-1)$ in this order and leaves at position p. A strand entering at position $j \geq 2$ meets only $(k, j-1)$ and leaves at position $j-1$.*

Proof. The i-th letter involves only positions $i, i+1$. From position 1 the strand is at position i before the i-th letter, by induction, so it is moved by every letter. From $j \geq 2$ the strand is untouched by letters $1, \ldots, j-2$ and by letters $j, \ldots, p-1$, and it is moved from j to $j-1$ by letter $j-1$. □

Let $D_a = \big((a+t,\, p-1-t)\big)_{t=0}^{p-2}$ (a *descent*) and $S_b = \big((b,i)\big)_{i=1}^{p-1}$ (a *sweep*); block indices are taken mod q.

Lemma 6. *Let* $\gcd(p,q) = 1$ *and* $a_j = a_0 + jp$. *Then* $\gamma(\Pi)$ *is the cyclic concatenation*

$$D_{a_0}\, S_{a_0+p-1}\, D_{a_1}\, S_{a_1+p-1} \cdots D_{a_{q-1}}\, S_{a_{q-1}+p-1}.$$

Every crossing occurs exactly once in a sweep and exactly once in a descent.

Proof. A strand entering block a at position p meets, by Lemma 5 applied $p-1$ times, the crossings $(a+t, p-1-t)$, $t = 0, \dots, p-2$, and enters block $a+p-1$ at position 1. There it meets S_{a+p-1} and enters block $a+p$ at position p. Since $\gcd(p,q) = 1$, the sweep blocks $a_j + p - 1$ are pairwise distinct mod q, and the descent start blocks a_j exhaust $\mathbb{Z}/q$. Each (k,i) lies in exactly one D_a, namely the one with $a \equiv k-(p-1-i)$. So the $2n$ entries contain every crossing twice, and the walk closes after q units. □

Proposition 7. *If* $p > q \geq 2$ *and* $\gcd(p,q) = 1$, *then* $\operatorname{maxrun}(\Pi) = 2q+p-3$, *and* $n - \operatorname{maxrun}(\Pi) = (q-1)(p-3)$.

Proof. We use $L \geq q$ throughout. The descent D_a has one entry in each of the L consecutive blocks $a, \dots, a+L-1$, so it meets every block. Hence $(*)$: no repetition-free run contains a complete sweep S_b together with a complete descent, since some (b,i) lies in both. By Lemma 6, S_b is preceded by D_{b-p+1} and followed by D_{b+1}, and D_a is preceded by S_{a-1} and followed by S_{a+p-1}. Reading off the coordinates $(a+t, p-1-t)$:

- (F1) the first entry of D_{b+1} in block b is $(b, p-q)$, at $t = q-1$;
- (F2) the last entry of D_{b-p+1} in block b is at $t = p-1-q$, and the $q-1$ entries after it avoid block b;
- (F3) the entries of D_a in block $a-1$ have second coordinate $\leq p-q$, attained at $t = q-1$;
- (F4) the entries of D_a in block $a+p-1$ have second coordinate $\geq q$, attained at $t = p-1-q$.

Let R be a repetition-free run. If R contains a complete sweep S_b, then by $(*)$ it lies in a tail of D_{b-p+1}, followed by S_b, followed by a head of D_{b+1}; by (F1) and (F2) the head and the tail have at most $q-1$ entries each, so $|R| \leq 2q+p-3$. If R contains a complete descent D_a but no complete sweep, then it lies in a tail of S_{a-1}, followed by D_a, followed by a head of S_{a+p-1}. A tail of S_{a-1} of length s is $\{(a-1,i) : i \geq p-s\}$, which by (F3) meets D_a iff $s \geq q$. Likewise, by (F4), a head of length h meets D_a iff $h \geq q$. So again $|R| \leq 2q+p-3$. If R contains no complete segment, it is a tail of one segment followed by a head of the next, of lengths $s, h \leq L-1$. By (F1), resp. (F4), $s \geq q$ and $h \geq q$ would produce a repetition, so $|R| \leq (q-1)+(L-1) < 2q+p-3$.

The run consisting of the last $q-1$ entries of D_{b-p+1}, then S_b, then the first $q-1$ entries of D_{b+1} is repetition-free: the descent pieces avoid block b by (F1), (F2), and they lie in two different descents ($b-p+1 \not\equiv b+1$

$\pmod q$, since $q \nmid p$ by $\gcd(p,q)=1$ and $q \geq 2$), which share no crossing by Lemma 6. Finally, $q(p-1)-(2q+p-3)=(q-1)(p-3)$. □

For $p > q \geq 3$ we have $p \geq 4$, so $\operatorname{maxrun}(\Pi) < n$, and Π is not in straight position. A shorter argument for this inequality alone: a window of $n \geq 3L$ consecutive entries contains two adjacent complete segments, one sweep and one descent, and $(*)$ applies.

For $q > p$ a descent no longer meets every block, and $(*)$ fails. We use a second description of $\gamma(\Pi)$. For $x \in \mathbb{Z}/pq$ let the *chunk* C_x be the single crossing $\big(x \bmod q,\ p-1-(x \bmod p)\big)$ if $x \not\equiv p-1 \pmod p$, and the sweep $S_{x \bmod q}$ if $x \equiv p-1 \pmod p$. Call x a *single* or a *sweep point* accordingly; $x \bmod q$ is the block of x.

Lemma 8. *Let* $\gcd(p,q)=1$. *Then* $\gamma(\Pi)$ *is the cyclic concatenation* $C_0C_1\cdots C_{pq-1}$. *Two singles carry the same crossing only if they are equal. Each block* k *has exactly one sweep point, and the crossing* (k,i) *occurs in its sweep and in the single* x *with* $x \equiv k \pmod q$, $x \equiv p-1-i \pmod p$.

Proof. Take $a_0 = 0$ in Lemma 6. The j-th unit $D_{jp}S_{jp+p-1}$ is $C_{jp}\cdots C_{jp+p-2}C_{jp+p-1}$, since D_{jp} has the entry $(jp+t,\ p-1-t)$ at $t = x-jp = x \bmod p$ for $x = jp+t$, $0 \leq t \leq p-2$. The units $j = 0,\dots,q-1$ give $x = 0,\dots,pq-1$. The other statements follow from the Chinese remainder theorem, as $x \in \mathbb{Z}/pq$ is determined by $(x \bmod q,\ x \bmod p)$. □

Proposition 9. *Let* $q > p \geq 3$, $\gcd(p,q)=1$, *and* $q = bp+r$ *with* $b \geq 1$, $1 \leq r \leq p-1$. *Then* $\operatorname{maxrun}(\Pi) = 2q+p-2b-3$, *and this is less than* $n = q(p-1)$.

Proof. Let R be a repetition-free run. It has at most n entries, so it meets at most $n < pq$ chunks; we lift the chunks it meets to an interval $I = [u,v]$ of integers with $v-u < pq$, so that distinct elements of I are distinct mod pq. All chunks C_x, $u < x < v$, lie completely in R; C_u and C_v may be incomplete only if they are sweeps, in which case R contains a tail of C_u of length $e \leq L-1$ and a head of C_v of length $h \leq L-1$. A sweep point $x \in I$ is *complete* if $C_x \subseteq R$. Sweep points are the $x \equiv p-1 \pmod p$. Two facts follow from Lemma 8.

(R1) If $x \in I$ is a complete sweep point, no other $y \in I$ has $y \equiv x \pmod q$ (it would be a single of the same block, whose crossing lies in C_x). Hence $I \subseteq [x-q+1,\ x+q-1]$.

(R2) If v is an incomplete sweep point and $v-q \in I$, then $h \leq r-1$: the single $v-q \equiv p-1-r \pmod p$ carries $(v \bmod q,\ r)$, and the head of $S_{v \bmod q}$ of length h is $\{(v \bmod q, i) : i \leq h\}$. Likewise, if u is an incomplete sweep point and $u+q \in I$, then $e \leq r-1$: the single $u+q \equiv r-1 \pmod p$ carries $(u \bmod q,\ p-r)$, and the tail of length e is $\{(u \bmod q, i) : i \geq p-e\}$.

Let f be the number of complete sweep points, $x_1 < \cdots < x_f$, so $x_j = x_1+(j-1)p$.

$f = 0$. Then I contains no sweep point except possibly u and v, so it lies between two consecutive sweep points, and $|R| \le (p-1)+2(p-2) = 3p-5$. This includes the case $u = v$, in which R is a sub-run of the single chunk C_u, proper if C_u is a sweep, so $|R| \le p-2$. As $q - b = b(p-1) + r \ge p$, this is less than $2q + p - 2b - 3$.

$f \ge 1$. Incomplete sweeps can only be $u = x_1 - p$ and $v = x_f + p$. Let σ_L and σ_R be the numbers of singles of I in $(x_1 - p, x_1)$ and in $(x_f, x_f + p)$; both are at most $p-1$, and I has $(f-1)(p-1)$ further singles, between x_1 and x_f. Thus $|R| = fL + (f-1)(p-1) + (\sigma_L + h) + (\sigma_R + e)$, with $h = 0$ (resp. $e = 0$) if v (resp. u) is not an incomplete sweep point.

$f \ge b+1$. Then $fp \ge (b+1)p = q + (p - r) > q$. By (R1) for x_1 and x_f, $I \subseteq [x_f - q + 1,\, x_1 + q - 1]$, an interval of $2q - 1 - (f-1)p$ integers. Since $x_f - x_1 + p = fp > q$, it contains neither $x_1 - p$ nor $x_f + p$, so all chunks of R are complete and $|R| = |I| + f(L-1) \le 2q - 1 - (f-1)p + f(p-2) = 2q + p - 1 - 2f \le 2q + p - 2b - 3$.

$f = b$. We show $\sigma_L + h \le p + r - 2$. If v is not an incomplete sweep point, $\sigma_L + h \le p - 1$. Otherwise $v = x_b + p$ and $v - q = x_1 + bp - q = x_1 - r$. If $x_1 - r \in I$, then $h \le r - 1$ by (R2). If not, $u > x_1 - r$, so $\sigma_L \le r - 1$, while $h \le p - 2$. In all cases $\sigma_L + h \le p + r - 2$. In the same way, with $u + q = x_1 - p + q = x_b + r$, $\sigma_R + e \le p + r - 2$. Hence $|R| \le (2b-1)(p-1) + 2(p + r - 2) = 2(bp + r) + p - 2b - 3 = 2q + p - 2b - 3$.

$1 \le f \le b - 1$. Then $|R| \le (2f+1)(p-1) + 2(p-2) \le (2b-1)(p-1) + 2(p-2) = 2q + p - 2b - 3 - 2r$.

So $\operatorname{maxrun}(\Pi) \le 2q + p - 2b - 3$. For equality let x_0 be a sweep point and let R consist of the complete chunks C_x, $x \in I = [x_0 - r + 1,\, x_0 + q - 1]$. The sweep points of I are $x_0 + jp$, $0 \le jp \le q - 1$, i.e. $j = 0, \dots, b$ (as $p > r - 1$ and $(b+1)p > q$). They lie in different blocks, since $0 < jp - j'p \le bp < q$ for $j > j'$. Two singles never carry the same crossing (Lemma 8). A single $y \in I$ of the block of $x_0 + jp$ would satisfy $|y - x_0 - jp| \ge q$, but $I \subseteq [x_0 + jp - q + 1,\, x_0 + jp + q - 1]$ because $jp \le bp = q - r$. So R is repetition-free, and $|R| = (q + r - 1) + (b+1)(L-1) = 2q + p - 2b - 3$.

Finally $n - (2q + p - 2b - 3) = q(p-3) - p + 2b + 3$, which is $2b > 0$ for $p = 3$, and positive for $p \ge 4$ since $q(p-3) \ge q > p$. □

For $p > q$ we have $\lfloor q/p \rfloor = 0$, and the formula $2q + p - 2b - 3$ reduces to that of Proposition 7.

Remark 10. Owad's walk in the proof of [Ow2, Theorem 3.3] (arXiv version v2, p. 5; $W(n, m)$ with $n = p$ strands and $m = q$ blocks, $m = bn + r$) is, in the notation above, the run of the equality part of the proof of Proposition 9: forward from a sweep through the next $q - 1$ chunks, and backward through $r - 1$ singles. Two points of the text there, stated as observations: the sum $2b(n-1) + (n-1) + 2(r-1)$ given there equals $2m + n - 2b - 3$, one less than the printed simplification $2m + n - 2(b+1)$; and that no other starting position gives more is stated with reference to the symmetry of the braid.

Proposition 9 gives the value $2m + n - 2b - 3$ and proves the maximality. Owad's conclusion is not affected, as both values are less than $m(n-1)$.

4. Uniqueness of the minimal diagram

For a diagram D on $S^2 \subset S^3$ let $\lambda(D) \subset S^3$ be the link it represents (the realised diagram of [St, §5]).

Proposition 11. *Let $p, q \geq 3$, $\gcd(p,q) = 1$. Every diagram of $Th(p,q)$ with $(p-1)q$ crossings has a shadow that is homeomorphic, by a homeomorphism of S^2 of one of the two orientations, to the shadow of Π.*

Proof. Let D' be such a diagram. It is reduced: removing a nugatory crossing would give a diagram with fewer than $c(Th(p,q))$ crossings. $Th(p,q)$ is prime [DS, §4.2], so D' is alternating by Murasugi [Mu, Theorems 2 and 3]: a connected proper alternating projection has $\operatorname{span} V = c$ (this applied to Π), while a non-alternating projection of a prime link has $\operatorname{span} V$ less than its number of crossings. D' is prime as a diagram: a decomposing circle would split D' into two connected reduced alternating diagrams with at least one crossing each, both non-trivial by [Mu, Theorem 2], and $Th(p,q)$ would be composite. Π is reduced, alternating and prime as a diagram [St, §2 and Remark 3.3]. By the flyping theorem of Menasco and Thistlethwaite [MT, Main Theorem], in the form of [St, Theorem 5.6], the orientation-preserving homeomorphism of pairs $(S^3, \lambda(\Pi)) \to (S^3, \lambda(D'))$ given by an ambient isotopy (with D' oriented accordingly) is a composite of flypes $\lambda(D_0) \to \cdots \to \lambda(D_m)$ with $D_0 = \Pi$, $D_m = D'$. In [St] it is shown that the Tait graph of Π is simple and 3-connected [St, Lemma 3.2], so every circle meeting Π transversally in four points away from the crossings has at most one crossing on one side [St, Lemma 3.1]; Π has no bigons, every face having degree 3, 4 or q [St, §2 and proof of Proposition 5.11]; and every flype on such a diagram is degenerate, with target shadow isomorphic to the source shadow or to its mirror [St, Lemmas 5.8 and 5.9]. The induction in the proof of [St, Proposition 5.11] shows that every D_i again has these properties and the shadow of Π or of its mirror. It proceeds forward from $D_0 = \Pi$ and does not use the end diagram, so it applies with $D_m = D'$. □

Remark 12. Owad's argument uses that the standard diagram is "the only diagram with the minimum number of crossing" [Ow2, proof of Theorem 3.3], stated as [Ow2, Lemma 3.2] up to planar isotopy; in the arXiv version v2, which is the version we read, a sketch of its proof is given. Read in $\mathbb{R}^2$, where straight position lives, the statement has to be taken on S^2: moving the point at infinity into a triangular face of Π gives a reduced alternating minimal diagram whose unbounded face is a triangle, while that of Π has q sides; for $q \geq 4$ the two are not planar isotopic. Proposition 11 is the S^2 form, reflections allowed, and by Lemma 4(a),(c) it is all our argument needs. It holds for all $p, q \geq 3$.

5. Proof of Theorem 1

Let $p, q \geq 3$, $\gcd(p,q) = 1$, $b = \lfloor q/p \rfloor$, and let D' be a diagram of $Th(p,q)$ with $c = (p-1)q$ crossings. By Proposition 11 and Lemma 4(c), $\mathrm{maxrun}(D') = \mathrm{maxrun}(\Pi)$. By Proposition 7 ($q < p$) and Proposition 9 ($q > p$) this equals $2q + p - 2b - 3 < c$. By Lemma 4(a), no planar isotopy puts D' in straight position, and by Lemma 4(b), $Th(p,q)$ is not perfectly straight. □

6. The case $q = 2$

Proof of Theorem 2. Let $p \geq 3$ be odd. By [Q2, Lemma 2], $Th(p,2)$ is a two-bridge knot. By [BM, Theorem 4], every two-bridge knot has a minimal diagram that is semi-meander, i.e. composed of two simple arcs [BM, (C1), (C3)]; minimal means that no diagram of the knot has fewer crossings [BM, §5]. By Lemma 4(a), a planar isotopy puts this diagram in straight position, with $c(Th(p,2))$ crossings, and by Lemma 4(b) $Th(p,2)$ is perfectly straight. □

Corollary 3 follows from Theorems 1 and 2. Theorem 2 agrees with Owad's table: $Th(3,2) = 4_1$ and $Th(5,2) = 8_{12}$ [DS, Table 1] have str = 4 and str = 8 [Ow1, Table 1].

Remark 13. For $q = 2$ the standard diagram has bigons and its Tait graph has parallel edges, and the analogue of Proposition 11 for $q = 2$ fails at least for $Th(5,2)$: by Proposition 7, $\mathrm{maxrun}(\Pi) = p+1$, which is $6 < 8$ for $p = 5$, yet $\mathrm{str}(8_{12}) = 8$ [Ow1, Table 1], so by Lemma 4 some minimal diagram of 8_{12} has a shadow different from that of Π. For $Th(3,2) = 4_1$ the standard diagram itself is in straight position after a planar isotopy, by Lemma 4(a), since $\mathrm{maxrun}(\Pi) = p+1 = 4 = n$. For $p \geq 7$ we have not examined minimal diagrams other than Π.

7. Checks

A first script computes $\gamma(\Pi)$ in three ways: (A) braid simulation; (B) a straight-ahead walk on the PD code that SnapPy's module spherogram [SnapPy] produces from the braid; (C) the formula of Lemma 6. (A) and (C) rest on the same reading of the braid; (B) is independent of it. The three sequences agree up to relabelling, rotation and reversal for $n \leq 150$; for larger n their maxruns were compared, and they agree. For all $p, q \leq 30$ with $\gcd(p,q) = 1$, $p \geq 3$, $q \geq 2$, it confirms $\mathrm{maxrun}(\Pi) = 2q + p - 3$ in all 234 cases with $3 \leq q < p$ and all 14 cases with $q = 2 < p$, and finds $\mathrm{maxrun}(\Pi) < n$ in all 234 cases with $q > p$. A second script, without external packages, compares the braid simulation with the chunk sequence of Lemma 8 (equal as cyclic sequences of labelled crossings in all 482 cases above) and confirms $\mathrm{maxrun}(\Pi) = 2q + p - 2\lfloor q/p \rfloor - 3$ in all of them; for the 234 cases with $q > p$ it confirms that the run of the equality part of Proposition 9 is repetition-free of that length, and that Owad's printed

count exceeds maxrun(Π) by exactly 1. As a calibration, $Th(4,3) = 9_{40}$ [DS, Table 1] is the only knot with $3 \le q < p$ and at most ten crossings. Owad's exhaustive table gives $\mathrm{str}(9_{40}) = 12 > 9$ [Ow1, Table 1], in agreement with Theorem 1; there maxrun(Π) = 7.

Tool use. This work was prepared with substantial use of Claude Code (Anthropic), an agentic AI system based on large language models, working together with the author.

- *Mathematics.* The problem was selected by the author from a list of open problems compiled with Claude Code from the survey [DS]. The proofs were worked out by Claude Code under the author's direction.
- *Writing and typesetting.* The text and the LaTeX source were written by Claude Code, in English. The note contains no figures.
- *References.* Literature was searched and sourced with Claude Code. Quotations and statements from [Ow1], [Ow2], [DS] and [BM] were checked against the arXiv versions named in the bibliography, those from [St] and [Q2] against their sources, and those from [Mu] against an OCR scan of the journal article. [MT] is used in the form stated in [St, Theorem 5.6]. The journal versions of [Ow2] and [BM] were not read; their bibliographic data were checked against Crossref.
- *Code.* The checks of the last section were written and run by Claude Code in Python: the environments are Python 3.11.15 with SnapPy 3.3.2 and spherogram 2.4.1 for the first script and Python 3.13.12 for the second, as installed when probed after the runs (file `VERSIONS.txt` next to the outputs).
- *Checking.* The draft was read in full by separate Claude Code agents in two adversarial cold reads of the full text, followed by diff reads of the two later revisions, and once more in full on 24 October 2026; their findings were checked and incorporated where they held.

The author takes full responsibility for all contents.

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