THE SYMMETRY GROUP OF TURK'S HEAD KNOTS

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Abstract. For $p, q \geq 3$ with $\gcd(p, q) = 1$ we show that the symmetry group of the Turk's head knot $Th(p, q)$, the closure of the braid $(\sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\cdots\sigma_{p-1}^{\pm 1})^q$, is the dihedral group of order $4q$. This proves Conjecture R of Di Prisa and Şavk for $q \geq 3$; the case $p = 3$ is due to Sakuma and Weeks. The argument is diagrammatic: by the flyping theorem of Menasco and Thistlethwaite, the symmetries of a knot with a prime, reduced alternating diagram without non-trivial flypes are the symmetries of the diagram (we write out a proof for the diagrams at hand), and we show that the standard diagram of $Th(p, q)$ has no non-trivial flypes and exactly $4q$ symmetries. The chirality of $Th(p, q)$ for even p, which also follows from a theorem of Murasugi, is visible in the group. We compare the argument with the geometric one of Sakuma and Weeks for $p = 3$.

1. Introduction

The Turk's head knot $Th(p, q)$ is the closure of the p-strand braid

$$\beta_p^q, \qquad \beta_p = \sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\cdots\sigma_{p-1}^{\pm 1}.$$

It is a knot if and only if $\gcd(p, q) = 1$. The survey of Di Prisa and Şavk [DS] collects what is known about these knots and poses a list of open problems. The symmetry group of a knot is defined there as the mapping class group of the knot exterior [DS, §5.2], and for $p = 3$ it is known from the work of Sakuma and Weeks [SW95, Proposition I.2.5] (quoted as Theorem M in [DS]). For p, q both odd, a result of Kodama and Sakuma implies that the group is dihedral of some order [DS, §6.3]. The survey states:

> **Conjecture R** [DS, §6.3]. *Suppose* $\gcd(p, q) = 1$. *Then the Turk's head knot* $Th(p, q)$ *has symmetry group given by* $\mathrm{Sym}(S^3, Th(p, q)) \cong D_{2q}$.

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Here D_n denotes the dihedral group with $2n$ elements, so D_{2q} has order $4q$. The survey assumes $p \geq 3$ and $q \geq 2$ throughout [DS, §4.1]. We prove the case $q \geq 3$; for $q = 2$ the diagram has bigons, and that case is not treated here.

Theorem 1.1. *Let $p \geq 3$, $q \geq 3$ with $\gcd(p, q) = 1$. Then $\mathrm{Sym}(S^3, Th(p, q)) \cong D_{2q}$, the dihedral group of order $4q$.*

Corollary 1.2. *Let $p \geq 3$, $q \geq 3$ with $\gcd(p, q) = 1$. For p even, $Th(p, q)$ is chiral. For p odd, $Th(p, q)$ is amphichiral.*

Both statements are known. The amphichirality for odd p is in [DS, §4.4]; it reappears here as an explicit element of the group. The chirality for even p follows from [Mur87, Corollary 5]: a proper alternating projection of an amphichiral alternating knot has an even number of crossings, and $(p-1)q$ is odd for p even. Here it is read off the group.

Method. The proof has three parts.

(A) *From the knot to the diagram* (Section 5). For a reduced alternating diagram without non-trivial flypes, every symmetry of the knot is isotopic to a symmetry of the diagram, and distinct symmetries of the diagram stay distinct. Surjectivity rests on the Main Theorem of Menasco and Thistlethwaite [MT93] in its form for homeomorphisms of pairs; injectivity rests on Borel's theorem on finite group actions, as restated in [CWY]. Menasco and Thistlethwaite note this consequence of their theorem [MT91, pp. 406–407], and Quach Hongler and Weber state the reduction [QHW, p. 4]; we give a proof for the diagrams at hand, for both orientations.

(B) *The diagram has no non-trivial flypes* (Section 3). Its Tait graph is simple and 3-connected, so every circle meeting the diagram in four points has at most one crossing on one side.

(C) *The diagram has exactly $4q$ symmetries, forming D_{2q}* (Section 4), by an explicit count.

Parts (B) and (C) are elementary. Part (A) uses the flyping theorem as a black box together with standard facts about hyperbolic knot exteriors, and four lemmas proved here (Lemmas 5.7, 5.8, 5.9 and 5.10).

For $p = 3$, Sakuma and Weeks obtain the group geometrically, from the canonical decomposition of the knot complement. Section 7 compares the two arguments.

2. The diagram

Let Π be the closure of β_p^q drawn in an annulus, with angle x and radius y; the strand positions ("levels") are $1, \ldots, p$ from inside out.

Crossings. The crossings are $c(i, k)$ with ring $i \in \{1, \ldots, p-1\}$ (between levels i and $i+1$) and $k \in \mathbb{Z}_q$; in the braid picture $c(i, k)$ sits at angle $\left(k + \frac{i-1}{p-1}\right)\frac{2\pi}{q}$.

Edges. Along level j the crossings met are, in angular order,

- $j = 1$: $c(1,k) \to c(1,k+1)$;
- $2 \le j \le p-1$: $c(j-1,k) \to c(j,k) \to c(j-1,k+1) \to c(j,k+1)$;
- $j = p$: $c(p-1,k) \to c(p-1,k+1)$.

So the edges of the 4-valent map M underlying Π are $\{c(1,k), c(1,k+1)\}$ on level 1; $\{c(j-1,k), c(j,k)\}$ and $\{c(j,k), c(j-1,k+1)\}$ on level j, $2 \le j \le p-1$; and $\{c(p-1,k), c(p-1,k+1)\}$ on level p.

Half-edges. At $c(i,k)$ the four half-edges are EL, EU, WU, WL: east or west by angle, lower or upper by level i or $i+1$. This is their counter-clockwise order in the chart with x to the right and the level upwards (the cylinder chart of Section 5.2); in the annulus, with x increasing counter-clockwise and the levels outwards, it is clockwise.

Crossing data. The diagram is alternating. The strand that rises (WL–EU) is the over-strand exactly when i is odd; this is the sign of $\sigma_i^{(-1)^{i+1}}$. (With the opposite braid convention everything below holds with "odd" and "even" exchanged.)

Faces. F_0 is the inner disc and F_p the outer face. For $1 \le i \le p-1$, $F(i,k)$ is the region of ring i between $c(i,k)$ and $c(i,k+1)$. Counting crossings on the boundary: $\deg F_0 = \deg F_p = q$; rings 1 and $p-1$ consist of triangles, each with one crossing of the neighbouring ring on one side; rings $2, \dots, p-2$ consist of quadrilaterals, each with one crossing of ring $i-1$ and one of ring $i+1$. There are $2 + q(p-1)$ faces, which is $c+2$ with $c = q(p-1)$. Faces in ring i share edges only with faces in rings $i \pm 1$ (an edge is an arc of a level), so colouring by the parity of i is the checkerboard colouring.

Primeness. $Th(p,q)$ is prime as a knot [DS, §4.2]; it is hyperbolic [DS, Theorem I], after Meyerhoff–Ruberman [MR90] and Menasco [Men84]. The diagram is alternating by construction and reduced by Lemma 3.2 (the Tait graph has no loops and, being 3-connected, no bridges); it is prime as a diagram by Remark 3.3 below.

3. The diagram is polyhedral

A *Haseman circle* of a diagram is a circle meeting it transversally in four points away from the crossings [QHW].

Lemma 3.1. *Let* Π *be a reduced alternating projection whose Tait graph* G *(one colour class of faces) is simple and* 3*-connected. Then every Haseman circle of* Π *bounds, on one side, a disc containing at most one crossing. Hence* Π *equals its polyhedral part.*

Proof. A Haseman circle γ crosses four edges, so it passes through four face-arcs. Consecutive face-arcs lie in faces on opposite sides of an edge, so their colours alternate: black B_1, white, black B_2, white.

Consider one side Δ of γ and suppose Δ contains no black face entirely. Every crossing in Δ has its two black corners in black faces that meet Δ, hence in B_1 or B_2. That crossing is therefore a Tait edge joining B_1 to B_2,

or a loop at B_1 or B_2. Loops are nugatory crossings, excluded because Π is reduced. Two crossings joining B_1 to B_2 would be parallel Tait edges, excluded because G is simple. So such a side contains at most one crossing.

Now suppose both sides contain a black face entirely. If $B_1 \neq B_2$, removing B_1 and B_2 disconnects G, contradicting 3-connectivity. If $B_1 = B_2$, removing that one vertex already disconnects G.

Hence one side contains at most one crossing. A circle around one crossing bounds a singleton. A circle with no crossing on one side is compressible and is excluded by Rule 2 of [QHW]. So the minimal Conway family consists of the circles around single crossings, and the rest is one diagram all of whose Haseman circles are boundary parallel, a basic diagram: Π is polyhedral. □

Lemma 3.2. *For $p \geq 3$, $q \geq 3$ the Tait graph of $Th(p, q)$ on the even-ring faces is simple and 3-connected.*

Proof. At the crossing $c(i, k)$ the two faces of ring i are opposite corners, and $F(i-1, k)$ and $F(i+1, k-1)$ are the other two corners; read $F(i-1, \cdot)$ as F_0 when $i = 1$, and $F(i+1, \cdot)$ as F_p when $i = p-1$. Take the even rings as the colour class. Then:

- a crossing with i even gives a Tait edge $F(i, k-1)$–$F(i, k)$, so each even ring $2 \leq i \leq p-1$ is a q-cycle;
- a crossing with i odd gives an edge $F(i-1, k)$–$F(i+1, k-1)$, a bijection between consecutive even rings respecting their cyclic order;
- F_0 (ring 0) is a hub joined once to each face of ring 2;
- if p is even, F_p is a second hub joined once to each face of ring $p-2$.

So G is a hub, then a stack of prisms $C_q \times P_m$ ($m \geq 1$ cycles), then a second hub if p is even. This is the graph of a 3-polytope: for p odd a stack of prisms with a pyramid on one end (a pyramid when $p = 3$), for p even a stack of prisms with a pyramid on each end (a bipyramid when $p = 4$). By Balinski's theorem [Bal61] G is 3-connected. It is simple because $q \geq 3$. □

Remark 3.3. Π is prime as a diagram. Suppose a circle met Π in two points and had crossings on both sides. It passes through one black and one white face. A side containing no black face entirely has only crossings that are Tait loops at that black face, i.e. nugatory crossings, which are excluded. So both sides contain black faces, and the black face it passes through is a cut vertex of the Tait graph, contradicting Lemma 3.2.

4. The symmetries of the diagram

Let $\mathrm{Sym}(\Pi)$ be the group of pairs (g, ε), where g is a homeomorphism of S^2 preserving Π up to isotopy (made precise as an isometry of a symmetric drawing in Section 5), and $\varepsilon = \pm 1$ is such that g sends over-strands to over-strands ($\varepsilon = +1$) or to under-strands ($\varepsilon = -1$).

Lemma 4.1 (upper bound)**.** $|\mathrm{Sym}(\Pi)| \leq |\mathrm{Aut}(M)| \leq 4q$.

Proof. For fixed g, ε is determined: g sends the over-strand of one crossing either to an over-strand or to an under-strand. So $|\operatorname{Sym}(\Pi)| \le |\operatorname{Aut}(M)|$, the group of map automorphisms of either orientation; and g is determined by its action on the crossings, since three crossings of ring 1 lie on the circle of latitude $h = 1 \neq p/2$, which is not a great circle, and an isometry of S^2 fixing three points not on one great circle is the identity. $\operatorname{Aut}(M)$ acts freely on flags (vertex, edge, face), because M is connected.

The pair $\{F_0, F_p\}$ is $\operatorname{Aut}(M)$-invariant:

- $q \ge 5$: F_0 and F_p are the only faces of degree q.
- $q = 4$ (so p is odd): the degree-4 faces are F_0, F_p and the quadrilaterals of rings $2, \dots, p-2$. F_0 and F_p border only triangles (rings 1 and $p-1$). A quadrilateral in ring i borders rings $i-1$ and $i+1$, and since $p \neq 4$ at least one of these is a quadrilateral ring.
- $q = 3$ (so $3 \nmid p$, $p \ge 4$): the triangles are F_0, F_p and rings 1 and $p-1$. F_0 and F_p border only triangles. A triangle of ring 1 borders F_0 and two faces of ring 2, which are quadrilaterals because $p \ge 4$; symmetrically, a triangle of ring $p-1$ borders F_p and two quadrilaterals of ring $p-2$.
- For $p = 3$ there are no rings of quadrilaterals; then $q \ge 4$, and F_0, F_p are the only q-gons.

Hence $\operatorname{Aut}(M)$ maps flags of $F_0 \cup F_p$ to flags of $F_0 \cup F_p$. There are $2q + 2q$ of them, and freeness gives $|\operatorname{Aut}(M)| \le 4q$. □

All indices k below are in $\mathbb{Z}_q$. Three kinds of maps:

map	on crossings	on half-edges	S^2	ε
ρ, rotation by one period	$c(i,k) \mapsto c(i,k+1)$	identity	$+$	$+1$
σ, inside-out, angle reversed	$c(i,k) \mapsto c(p-i,-k)$	$EL \leftrightarrow WU$, $EU \leftrightarrow WL$	$+$	$(-1)^p$
μ_a, reflection, $a \in \mathbb{Z}_q$	$c(i,k) \mapsto c(i,a-(i-1)-k)$	$EL \leftrightarrow WL$, $EU \leftrightarrow WU$	$-$	-1

Lemma 4.2 (lower bound)**.** *ρ, σ and all μ_a are automorphisms of M with the stated sign ε, and the $4q$ elements ρ^k, μ_a, $\sigma\rho^k$, $\sigma\mu_a$ are distinct. Hence* $|\operatorname{Sym}(\Pi)| = 4q$.

Proof. ρ is clear. σ sends level j to level $p+1-j$: the level-1 edge $\{c(1,k), c(1,k+1)\}$ goes to the level-p edge $\{c(p-1,-k), c(p-1,-k-1)\}$ and conversely, and for $2 \le j \le p-1$ it maps $\{c(j-1,k), c(j,k)\} \mapsto \{c(p-j+1,-k), c(p-j,-k)\}$, an edge of the first type on level $p+1-j$, and $\{c(j,k), c(j-1,k+1)\} \mapsto \{c(p-j,-k), c(p-j+1,-k-1)\}$, an edge of the second type. The half-edge order $(EL, EU, WU, WL) \mapsto (WU, WL, EL, EU)$ is a cyclic shift, so σ preserves the orientation of S^2. The rising strand $\{WL, EU\} \mapsto \{EU, WL\}$ is again rising, now in ring $p-i$, so over-strands

go to over-strands if and only if $i \equiv p - i \pmod 2$, i.e. if and only if p is even.

μ_a fixes the rings. For $c(i,k) \mapsto c(i, a_i - k)$ to send edges to edges one needs $a_{j-1} = a_j + 1$; constant a_i fails. So $a_i = a - (i-1)$. It reverses the half-edge order, so it reverses the orientation of S^2, and it sends the rising strand to the falling one, so $\varepsilon = -1$.

The four families are told apart by whether F_0 is fixed and by the orientation on S^2; within a family, the image of $c(1,0)$ differs. Each of them is realised by an isometry of the symmetric drawing (Section 5.2). Together with Lemma 4.1, $\mathrm{Sym}(\Pi) = \{\rho^k, \mu_a, \sigma\rho^k, \sigma\mu_a\}$ has exactly $4q$ elements. □

Lemma 4.3. $\mathrm{Sym}(\Pi) \cong D_{2q}$.

Proof. Let $\tau = \sigma\mu_a$. Then $\tau(c(i,k)) = c(p-i, k+i-1-a)$ and $\tau^2 = \rho^{p-2-2a}$. If q is odd, choose $2a \equiv p-3 \pmod q$; then $\tau^2 = \rho$. If q is even, p is odd, $p-3$ is even, and $2a \equiv p-3 \pmod q$ is solvable; again $\tau^2 = \rho$, of order q. Moreover $\tau^q \neq 1$: σ exchanges F_0 and F_p and μ_a fixes both, so odd powers of τ exchange them (q odd), and $\tau^q = \rho^{q/2} \neq 1$ (q even). So τ has order $2q$. Further, μ_0 is an involution and lies outside $\langle\tau\rangle$, because it fixes F_0 and reverses the orientation of S^2, while the elements of $\langle\tau\rangle$ fixing F_0 are the rotations ρ^k. Finally $\mu_0\tau\mu_0 = \tau^{-1}$: both send $c(i,k)$ to $c(p-i, k+a+i+1-p)$. Hence $\mathrm{Sym}(\Pi) = \langle\tau\rangle \rtimes \langle\mu_0\rangle \cong D_{2q}$. □

Remark 4.4 (chirality). An element (g,ε) acts on S^3 preserving orientation if and only if $\mathrm{sign}(g)\cdot\varepsilon = +1$ (Section 5). For p even no element of the table, hence none of $\mathrm{Sym}(\Pi)$, has $\mathrm{sign}(g)\cdot\varepsilon = -1$. For p odd, σ has $\mathrm{sign}(g) = +$ and $\varepsilon = -1$. With Theorem 1.1 this gives Corollary 1.2.

5. From the knot to the diagram

5.1. **Conventions.** $\mathrm{Sym}(S^3, K)$ is defined in [DS, §5.2] as the mapping class group of the knot exterior E. We prove the statement both for the pair (S^3, K) and for E (Section 5.14). Let $K = Th(p,q)$ with $p, q \geq 3$, $\gcd(p,q) = 1$; K is hyperbolic (Section 2).

Category. As in [MT93], links are tame and homeomorphisms of pairs are piecewise linear (they use this on p. 125: "since f is piecewise linear"); the proofs of Lemmas 5.7 and 5.10 use transversality and the Alexander trick in that category. Isometries of the round model enter in the same way as the rigid rotations and isometric crossing balls of [MT93, p. 115] and [MT91, p. 405]. We rely on the classical fact that for tame links in S^3 the classification of homeomorphisms of pairs up to pairwise isotopy does not depend on the choice between the topological, piecewise linear and smooth categories, and do not reprove it.

Model of S^3. $S^3 = \{(y,z) \in \mathbb{R}^3 \times \mathbb{R} : |y|^2 + z^2 = 1\}$, with projection sphere $S^2 = \{z = 0\}$. A point "over $u \in S^2$ at height z" is $(\sqrt{1-z^2}\,u, z)$. A pair (g,ε) with $g \in O(3)$, $\varepsilon = \pm 1$ acts by $(y,z) \mapsto (gy, \varepsilon z)$; it preserves the orientation of S^3 if and only if $\det(g)\cdot\varepsilon = +1$. Stereographic projection

from $(0,1)$, $\mu(y,z) = y/(1-z)$, identifies S^3 with $\mathbb{R}^3 \cup \{\infty\}$, the ambient space of [MT93, p. 114]. It is the identity from S^2 onto the unit sphere $\{\|x\| = 1\}$ on which their diagrams lie [MT93, p. 115], and it maps the point over u at height z to $\sqrt{(1+z)/(1-z)}\,u$; so it carries the product structure (u,z) near S^2 to their radial one, with the increasing change of parameter $z \mapsto \sqrt{(1+z)/(1-z)}$, $0 \mapsto 1$. Being conformal, μ maps metric balls of S^3 whose closure does not contain $(0,1)$ to Euclidean balls.

Realised diagram $\lambda(\Pi)$, star model. Take the symmetric drawing of Section 5.2. Inside a small disc of spherical radius r around each crossing c, replace the drawing by the four geodesic rays from c to the points where the edges meet the circle of radius r. The over-strand is the union of two rays opposite in the cyclic order at c, lifted to height $z = +\beta(d)$, d the spherical distance to c; the under-strand is lifted to $z = -\beta(d)$, with $\beta(d) = \sqrt{r^2 - d^2}$. Up to pairwise isotopy this is the semicircle profile of the crossing balls of Menasco and Thistlethwaite [MT93, p. 114]: in the round metric the top point of the over-strand lies at distance $\arcsin r > r$ from c, so the strands lie on the boundary of the metric ball of radius r about c only after an isotopy. An isometry g preserving the drawing maps rays from c to rays from $g(c)$ and preserves d; so (g,ε) maps $\lambda(\Pi)$ to itself exactly when it sends over-strands to over-strands ($\varepsilon = +1$) or to under-strands ($\varepsilon = -1$).

Menasco and Thistlethwaite require crossing balls isometric to a "standard pair" whose two strands are semicircles on the boundary of the ball, in perpendicular planes [MT91, p. 405]; they project to perpendicular diameters. The map μ, followed by a fixed homeomorphism of pairs that has product form $k \times \mathrm{id}$ near the unit sphere, k orientation-preserving, carries the star model to theirs up to pairwise isotopy: k straightens each pair of opposite rays to a diameter and makes the two diameters perpendicular, and it is radial in each crossing disc (in exponential coordinates at c it maps rays to rays and preserves d), so the semicircle profile carries over, up to the change of height parameter. All statements below are about pairwise isotopy classes, so they transfer along it.

The classes $\mathcal{F}$ and $\mathcal{F}^{\pm}$. Fix a product neighbourhood $N \cong S^2 \times [-\zeta, \zeta]$ of S^2 containing all realised diagrams in question. $\mathcal{F}^{\pm}$ is the set of homeomorphisms of pairs that on N have the form $h_0 \times (z \mapsto \varepsilon z)$, with h_0 a homeomorphism of S^2 and $\varepsilon = \pm 1$; $\mathcal{F} \subset \mathcal{F}^{\pm}$ is the subset with $\mathrm{sign}(h_0) \cdot \varepsilon = +1$. Both are closed under composition, because N is common. In the terms of [MT93]:

- $\varepsilon = +1$ is *flat*. Their definition [MT93, p. 116]: g is flat if it is pairwise isotopic to a homeomorphism h with $h(N) = N$ and $h|_N = h_0 \times \mathrm{id}_{[1/2,3/2]}$, h_0 an orientation-preserving homeomorphism of S^2, where $N = \{1/2 \le \|x\| \le 3/2\} \subset \mathbb{R}^3 \cup \{\infty\}$. Their model is the unit sphere in $\mathbb{R}^3 \cup \{\infty\}$ with the radial product structure on N [MT93, pp. 114–116] (the suspension picture is in [MT91, p. 405]); μ carries our product structure to it, and a radial homeomorphism

$x \mapsto \phi(\|x\|)\, x/\|x\|$, ϕ increasing with $\phi(1) = 1$, carries $\mu(N)$ onto their N. Together with $k \times \mathrm{id}$ this fixed transfer carries maps of the form $h_0 \times \mathrm{id}$, h_0 orientation-preserving, to maps of the same form. So the definition itself gives product form on N.

- $\varepsilon = -1$ in $\mathcal{F}$ (so $\mathrm{sign}(h_0) = -1$) is $R \circ \mathrm{flat}$, R a rigid rotation through π about an axis in S^2.

Let $\Phi\colon \mathrm{Sym}(\Pi) \to \mathrm{Sym}(S^3, K)$ send (g, ε) to the class of the orthogonal map (g, ε) of Section 5.2. We show that Φ is bijective.

5.2. **A symmetric drawing.** Draw Π on the cylinder $S^1 \times (0, p)$, with angle $\theta \in \mathbb{R}/q\mathbb{Z}$ and height h, embedded equivariantly in the round S^2: $\theta \mapsto$ longitude $2\pi\theta/q$, $h \mapsto$ latitude, odd about $h = p/2$. The crossing $c(i,k)$ sits at $(\theta, h) = (k + \frac{i-1}{2},\, i)$. Edges are straight segments in (θ, h): on level j, $2 \le j \le p-1$, from $(k + \frac{j-2}{2}, j-1)$ to $(k + \frac{j-1}{2}, j)$ and on to $(k + \frac{j}{2}, j-1)$, a zig-zag with steps $\Delta\theta = \frac{1}{2}$; level 1 is the horizontal segment at $h = 1$ and level p the one at $h = p-1$. Two zig-zags in adjacent strips meet only at their common vertices $c(j,k)$, where a peak of the lower strip meets a valley of the upper one. The four rays there are $SE, NE, NW, SW = EL, EU, WU, WL$, in the order of the combinatorial model (at rings 1 and $p-1$ the rays along level 1, resp. level p, are horizontal, E and W; the cyclic order is the same); so this is a planar drawing of M. (Offsets $\frac{i-1}{2}$ instead of the angular $\frac{i-1}{p-1}$ of the braid picture make μ_a rigid.)

The generators are affine in (θ, h), of the form $\theta \mapsto \pm\theta + \mathrm{const}$ with $h \mapsto h$ or $h \mapsto p - h$, hence isometries of S^2: $\rho\colon \theta \mapsto \theta + 1$; $\mu_a\colon \theta \mapsto a - \theta$, h fixed, so that $c(i,k) \mapsto c(i, a-(i-1)-k)$ since $a - (k + \frac{i-1}{2}) = (a-(i-1)-k) + \frac{i-1}{2}$; and $\sigma\colon \theta \mapsto \frac{p-2}{2} - \theta$, $h \mapsto p - h$, so that $c(i,k) \mapsto c(p-i, -k)$, a rotation by π about an equatorial axis. They permute the vertices as in Section 4 and preserve the drawing, since affine maps send segments to segments. With ε as there they satisfy the over/under rule, so they preserve $\lambda(\Pi)$. Thus $G := \langle \rho, \sigma, \mu_a \rangle \cong \mathrm{Sym}(\Pi)$ is a group of $4q$ orthogonal maps of $(S^3, \lambda(\Pi))$.

5.3. **Injectivity.** Let E be the exterior of K, the complement of an open G-invariant regular neighbourhood, e.g. a small metric ϵ-neighbourhood (G acts by isometries; $\lambda(\Pi)$ has corners where the semicircles meet S^2). G acts on E and preserves ∂E. Since K is hyperbolic, E is irreducible and aspherical with incompressible torus boundary, and K is prime (a composite knot has an essential torus); $\pi_1(E)$ is torsion-free and non-elementary Kleinian, hence centreless.

Let $DE = E \cup_{\partial E} E$ be the double; G acts effectively on DE by the same map on both copies. DE is a closed aspherical 3-manifold (gluing aspherical spaces along a π_1-injective aspherical torus), and $\pi_1(DE) = \pi_1(E) *_{\mathbb{Z}^2} \pi_1(E)$. Since $\mathbb{Z}^2 \neq \pi_1(E)$, the centre of this amalgam lies in $\mathbb{Z}^2$ and is central in $\pi_1(E)$, hence trivial. Borel's theorem, in the restatement of Cappell, Weinberger and Yan [CWY, pp. 1–2]: if the fundamental group of a closed aspherical manifold is centreless, the finite groups that act effectively act as

outer automorphisms of the fundamental group. So $G \to \mathrm{Out}(\pi_1(DE))$ is injective. We use the theorem only in this restatement; [CWY, p. 10] trace it to Borel and attribute the proof for manifolds to Conner and Raymond.

Proposition 5.4. Φ *is injective.*

Proof. Suppose $g \in G$ is pairwise isotopic to the identity on (S^3, K). Then $g|_E$ is homotopic to the identity (restrict the isotopy to $S^3 \setminus K$ and compose with a deformation retraction $S^3 \setminus K \to E$), so g_* is inner on $A := \pi_1(E)$, say conjugation by γ, with base point $x_0 \in \partial E$, where g_* is formed using a path from x_0 to $g(x_0)$ inside ∂E (possible since g preserves ∂E). Then g_* maps the peripheral subgroup $P \cong \mathbb{Z}^2$ onto itself, so $\gamma P \gamma^{-1} = P$ meets P non-trivially. The peripheral subgroup of a hyperbolic knot group is malnormal (a maximal parabolic subgroup), hence $\gamma \in P$. On $\pi_1(DE) = A *_P A$, the double g_D acts as conjugation by the same $\gamma \in P$ on both factors, i.e. as conjugation by the image of γ; so g_D induces the trivial outer automorphism. By Borel, $g_D = \mathrm{id}$, so $g = 1$. □

This covers orientation-reversing elements as well. No orbifold theorem, no Mostow rigidity and no equivariant sphere or torus theorem is used.

5.5. **Surjectivity, orientation-preserving symmetries.** We use the definitions of [MT93, §1, pp. 115–116]. A *standard flype* [MT93, p. 115]: in the pattern of their Fig. 1.1(i) four balls S_x, S_A, S_y, S_B sit in a row; S_x is a crossing ball, S_y meets the link in two parallel segments, and the two left ends of S_x are joined to the two right ends of S_B by two outer arcs. The flype rotates the tangle in S_A rigidly through π about an axis in the projection plane, fixes S_B pointwise, and maps S_x and S_y into themselves by half-twists. A *flype* is $g_1 \circ f' \circ g_2$ with f' standard and g_1, g_2 flat [MT93, p. 116]; and "since the property of being flat depends only on the pairwise isotopy class of a homeomorphism, the same is true for the property of being a flype" [MT93, p. 116].

Theorem 5.6 (Menasco–Thistlethwaite [MT93, Main Theorem, p. 116])**.** *Let D_1 and D_2 be reduced, prime, oriented, alternating link diagrams and let there be an orientation-preserving homeomorphism of pairs $f\colon (S^3, \lambda(D_1)) \to (S^3, \lambda(D_2))$. Then f is a composite of flypes.*

The announcement [MT91, Theorem 1] ends "Then f is pairwise isotopic to a composite of flypes". The two forms are equivalent: if $f \simeq h = f_n \circ \cdots \circ f_1$, then $f = (f \circ h^{-1} \circ f_n) \circ f_{n-1} \circ \cdots \circ f_1$, and the first factor is pairwise isotopic to f_n, hence a flype by the invariance just quoted. [MT93] uses the isotopy form in the proof of Proposition 4.10 (p. 148).

Degenerate flypes [MT93, p. 116]: "if the tangle A of Figure 1.1 contains no crossing, then the standard flype determined by that figure is a flat homeomorphism". For S_B without crossings the paper speaks only of diagrams [MT93, p. 113]: "the distortion amounts merely to a rotation through π of the complete link diagram about an axis in the projection 2-sphere". So

the S_A-empty case of Lemma 5.9 is in [MT93]; the S_B-empty case is proved here.

Lemma 5.7 (trivial tangles are rigid rel boundary)**.** *Let B be a 3-ball and $\tau \subset B$ a trivial (rational) 2-string tangle. Every homeomorphism of pairs $f\colon (B,\tau) \to (B,\tau)$ with $f|_{\partial B} = \mathrm{id}$ is isotopic to the identity rel ∂B, through homeomorphisms of pairs.*

Proof. Let $D \subset B$ be a properly embedded disc separating the two strings, with $\partial D \subset \partial B \setminus \tau$. (1) $f(D)$ has the same boundary; isotope it rel ∂ to be transverse to D, so $D \cap f(D)$ consists of ∂D and closed curves. (2) Remove the closed curves by the innermost-disc argument in the handlebody $B \setminus N(\tau)$, which is irreducible; each innermost pair of discs bounds a ball missing τ. Then $D \cup f(D)$ is a sphere in the handlebody; it bounds a ball there, across which $f(D)$ is isotoped onto D. (3) Now $f(D) = D$; isotope f rel ∂D to be the identity on D (Alexander trick on the disc). (4) D cuts (B,τ) into two balls, each with one unknotted proper arc, and f is the identity on their boundaries. Identify each with (unit ball, diameter) and apply the conical Alexander trick $F_t(x) = t\, f(x/t)$; it preserves the diameter, which is a cone. □

Lemma 5.8 (only degenerate flypes)**.** *Let D be reduced, alternating, polyhedral (Lemma 3.1) and without bigons. Then in every standard-flype pattern on D the tangle in S_A or the tangle in S_B has no crossing.*

Proof. The active crossing x shares two edges with S_A (its right ends) and, via the two outer arcs, two edges with S_B (its left ends). If S_A contained exactly one crossing, it and x would bound a bigon between them; if S_B did, the bigon would be the outer face, bounded by the two outer arcs. So each of S_A, S_B has 0 or ≥ 2 crossings. The circle around $S_x \cup S_A$ meets D in four points; by Lemma 3.1 one side carries at most one crossing. The side $S_x \cup S_A$ carries x, and ≥ 3 crossings if S_A is non-empty. So S_A is empty, or S_B has ≤ 1 and hence 0 crossings. (An empty S_A or S_B is the 0-tangle, two parallel arcs; the ∞-tangle would make x nugatory, so D would not be reduced.) □

Lemma 5.9 (degenerate flypes lie in $\mathcal{F}$)**.** *In the situation of Lemma 5.8, with D_1 a diagram of a prime knot, every standard flype $f'\colon \lambda(D_1) \to \lambda(D_2)$ is pairwise isotopic to a map in $\mathcal{F}$.*

Proof. First the target D_2. In both cases below the shadow of D_2 is isomorphic to that of D_1 (a planar isotopy moving x across the empty S_A, or the reflection R with x moved across the empty S_B). So D_2 is a diagram with c crossings of the same knot type as D_1. By Murasugi [Mur87, Theorems 2 and 3], a connected proper alternating projection $\tilde{L}$ of an alternating link L has $\operatorname{span} V_L(t) = c(\tilde{L})$, and for a prime link every non-alternating projection has $\operatorname{span} V_L(t) < c(\tilde{L})$. So $\operatorname{span} V(K) = c$ from D_1, and since K is prime,

D_2 is alternating. (Murasugi notes after Theorem 3 that primeness is needed there.) The crossing-type arguments below use that D_2 alternates.

S_A empty. Let T be the closure of the complement of the S_B-ball, a ball containing S_x, S_A, S_y. Then $T \cap \lambda(D_1)$ is a 1-crossing tangle, hence rational. Inside the disc $T \cap S^2$, D_2 differs from D_1 only by the position of x relative to the two parallel arcs of S_A. Both are 1-crossing tangle diagrams in that disc with the same four boundary points, the same connections (the strands join opposite corners) and the same crossing type, which is forced by alternation with the crossings of S_B next to them. So there is a flat $h\colon \lambda(D_1) \to \lambda(D_2)$ that is the identity on the S_B-ball (a planar isotopy supported in $T \cap S^2$, times id). Then $h^{-1} \circ f'$ preserves $(T, T \cap \lambda(D_1))$ and is the identity on ∂T, since f' fixes S_B pointwise. By Lemma 5.7 it is isotopic to id rel ∂T. Hence $f' \simeq h$, which is flat. (This case is also stated in [MT93, p. 116].)

S_B empty. Here the text of [MT93] uses two pictures at once. The diagram lies on the unit sphere [MT93, p. 115], while the circles of Fig. 1.1 are "euclidean 2-spheres meeting the (extended) projection plane in their respective equators", and (i) of the definition is "a rigid rotation through π about an axis in the projection plane" [MT93, p. 115]. A rotation through π about a line preserves the unit sphere only if the line passes through 0, and then it does not exchange the two sides of the sphere, as a rotation about an axis in the projection plane does; so the definition has to be interpreted. We read it in the round model, where both clauses hold literally. The balls are metric balls centred on S^2, with centres on one great circle $C \subset S^2$ (the row of Fig. 1.1; a general pattern is reached by flat maps [MT93, p. 116]); they meet S^2 in their equators, since the reflection $(y, z) \mapsto (y, -z)$ in S^2 preserves them. The four points in which $\lambda(D_1)$, and likewise $\lambda(D_2)$, meets the boundary of each ball lie on its equator, spaced equidistantly and symmetrically about C, as in Fig. 1.1; on p. 143 of [MT93] the four points on the Conway circle of a normal tangle are likewise spaced equidistantly, there for convenience, while the symmetry about C is taken from Fig. 1.1. A rigid rotation through π of such a ball about an axis in the projection sphere is the restriction of an isometry $R = (g, -1)$ of S^3, with g the reflection of $\mathbb{R}^3$ in the plane through 0 containing that axis; R is the rotation of S^3 through π about the great circle in which this plane meets S^2. For rotations about C this reading is their Fig. 1.1 literally. Stereographic projection from a point of C outside the four balls and the link is conformal; it maps S^2 to a plane P and the reflection in S^2 to the reflection in P, hence the four balls to Euclidean balls centred on P and meeting it in their equators; and it maps a rotation R about C, an orientation-preserving involution fixing C and the centre of projection, to the rigid rotation through π about the line that is the image of C. The projection μ from $(0, 1)$, on the other hand, carries the round model to their unit sphere and product neighbourhood N (*Model of S^3* above); the two projections differ by a Möbius transformation of $\mathbb{R}^3 \cup \{\infty\}$. Any two standard flypes for one pattern are pairwise isotopic: the composite of one with the inverse of the other is pairwise isotopic to a

map that fixes every crossing ball pointwise and maps every complementary region into itself, and therefore to the identity [MT93, p. 115]. We take the round reading as the meaning of their definition; so it suffices to treat one standard flype in it, and we apply Theorem 5.6 with standard flypes in this reading. By (iii) of the definition f' maps the S_x-ball into itself, and D_2 has the pattern of their Fig. 1.1(ii); the two arcs of $\lambda(D_1)$ joining S_x to S_A lie outside the four balls, so their images join S_x to the two left end points of S_A. Hence the rotation of the S_A-ball maps its left pair of end points to itself; the four end points being equidistant and symmetric about C, the only axis in the projection plane through its centre doing so is C, the row axis. So the axis is the row axis through the centre of S_A, and f' and R agree on the S_A-ball. Let U be the closure of the complement of the S_A-ball; $U \cap \lambda(D_1)$ is again a 1-crossing tangle. In Conway notation $D_1 = N(1 + A)$ and $D_2 = N(A^h + 1)$, while $R(D_1) = N(1^h + A^h) = N(1 + A^h)$. Inside the disc $U \cap S^2$, $R(D_1)$ and D_2 are 1-crossing tangle diagrams with the same boundary points and connections (the crossing joined directly to one side of A^h and around the outer arcs to the other), and their crossing types agree, being forced by alternation with the non-empty A^h. So there is a flat $h' \colon \lambda(RD_1) \to \lambda(D_2)$ that is the identity on the S_A-ball. Then $(h'R)^{-1} \circ f'$ is the identity on the S_A-ball and preserves $(U, U \cap \lambda(D_1))$; by Lemma 5.7 it is isotopic to id rel ∂U. Hence $f' \simeq h' \circ R \in \mathcal{F}$. □

Lemma 5.10 (rigidity)**.** *If $\varphi, \varphi' \in \mathcal{F}^{\pm}$ both preserve $\lambda(\Pi)$ and induce the same automorphism of M with the same ε, they are pairwise isotopic.*

Proof. $\psi := \varphi'^{-1} \circ \varphi \in \mathcal{F}^{\pm}$ is $k_0 \times \mathrm{id}$ on N, with k_0 fixing every flag of M. (1) ψ preserves $\lambda(\Pi)$. Over each crossing disc the height $|z| = \beta(d)$ determines the distance d to c, β being strictly monotone; so k_0 maps each ray from c to a ray at the same distances, by flag preservation to itself. Hence $k_0 = \mathrm{id}$ on the rays. (2) Outside the crossing discs k_0 maps each edge of Π to itself with fixed endpoints; isotope it to the identity along the edges by $k_t \times \mathrm{id}$, keeping $k_t = \mathrm{id}$ on the rays inside the crossing discs. This preserves $\lambda(\Pi)$, because there $\lambda(\Pi) = \Pi$ at $z = 0$. (3) Now $k_0 = \mathrm{id}$ on Π and each face goes to itself; use the Alexander trick face by face rel boundary, $k_t \times \mathrm{id}$ fixing $\lambda(\Pi)$ pointwise. (4) Then $\psi \simeq \mathrm{id}$ on N by an isotopy that is a product on N; extend it over the two complementary cap balls by coning, since they contain no part of K. Now ψ is the identity on N and on the boundary of each cap, and the Alexander trick in each cap finishes. □

Proposition 5.11. *Every orientation-preserving element of* $\mathrm{Sym}(S^3, K)$ *is in the image of* Φ.

Proof. Let Φ_0 be an orientation-preserving symmetry of $(S^3, \lambda(\Pi))$. By Theorem 5.6, $\Phi_0 \simeq f_n \circ \cdots \circ f_1$ with flypes $f_i \colon \lambda(D_{i-1}) \to \lambda(D_i)$, $D_0 = D_n = \Pi$. Write $f_i = g_1 \circ f' \circ g_2$.

Claim: every D_i, and every diagram met in between, has underlying map isomorphic to M or to its mirror and is reduced, alternating, polyhedral and

bigon-free. It holds for $D_0 = \Pi$ by Lemmas 3.1 and 3.2 and because the faces of Π have degrees 3, 4 and $q \geq 3$ (Section 2); flat maps keep it; and by Lemma 5.8 each f' is degenerate, its target has shadow isomorphic to D (S_A empty) or to $R(D)$, a mirror map (S_B empty), and it is alternating by the Murasugi argument in Lemma 5.9. So Lemmas 5.8 and 5.9 apply at every step, each f_i is isotopic to a map in $\mathcal{F}$, and so is the composite: $\Phi_0 \simeq \varphi \in \mathcal{F}$ with $\varphi(\lambda(\Pi)) = \lambda(\Pi)$.

Near S^2, $\varphi = h_0 \times \varepsilon$. Since φ maps $\lambda(\Pi)$ to itself, h_0 maps Π to Π, inducing $\alpha \in \mathrm{Aut}(M)$, and ε obeys the over/under rule for α. By Section 4, $|\mathrm{Aut}(M)| \leq 4q$, ε is determined by α, and G realises $4q$ distinct automorphisms; so some $g \in G$ induces α with the same ε. By Lemma 5.10 with $\varphi' = g$, $\Phi_0 \simeq g$.

If Φ_0 reverses the orientation of K, apply Theorem 5.6 with $D_2 = \Pi$ reversed; the map M is the same. The maps of [MT93] respect the link orientation ("preserving orientations of graphs and ambient spaces", [MT93, p. 117]); their remark on reversibility [MT93, Corollary 1.1, pp. 116–117] is this use. □

5.12. **Surjectivity, orientation-reversing symmetries.**

Proposition 5.13. *Every orientation-reversing element of* $\mathrm{Sym}(S^3, K)$ *is in the image of* Φ.

Proof. Let Φ_0 reverse the orientation of S^3, and let $r = (\mathrm{id}, -1)$, the reflection in S^2, which lies in $\mathcal{F}^{\pm}$ with $\det \cdot \varepsilon = -1$. Then $r \circ \Phi_0 \colon \lambda(\Pi) \to \lambda(\Pi^*)$ preserves orientation, where Π^* is Π with all crossings switched (same map M). Theorem 5.6 with $D_2 = \Pi^*$ (reversed if $r \circ \Phi_0$ reverses the orientation of K, as in Proposition 5.11), the induction in the proof of Proposition 5.11 (with $D_n = \Pi^*$) and Lemma 5.9 give $r \circ \Phi_0 \simeq \varphi \in \mathcal{F}$. So $\Phi_0 \simeq r^{-1} \circ \varphi$, which near S^2 is $h_0 \times (-\varepsilon)$ with $\mathrm{sign}(h_0) \cdot (-\varepsilon) = -1$ and which maps $\lambda(\Pi)$ to itself. As before some $g \in G$ induces the same automorphism with the same sign $-\varepsilon$, and Lemma 5.10 (for $\mathcal{F}^{\pm}$) gives $\Phi_0 \simeq g$. □

5.14. **Pair and exterior.** By uniqueness of regular neighbourhoods and isotopy extension, a homeomorphism of the pair can be isotoped to preserve E, and a pairwise isotopy between two such homeomorphisms can be replaced by one preserving E throughout. So restriction gives well-defined maps $\mathrm{Sym}(\Pi) \to \mathrm{MCG}(S^3, K) \to \mathrm{MCG}(E)$. The composite is injective, since the proof of Proposition 5.4 works on E. It is surjective: a homeomorphism h of E sends the meridian to $\pm$ the meridian, since otherwise Dehn filling would give S^3 by a non-trivial surgery on the non-trivial knot K, which Gordon and Luecke [GL89] exclude; so h extends to a homeomorphism of the pair, which is isotopic to some $g \in G$ by Propositions 5.11 and 5.13, and h is isotopic to $g|_E$ by the same remark. Hence both mapping class groups are isomorphic to $\mathrm{Sym}(\Pi)$.

6. Proof of Theorem 1.1

By Propositions 5.4, 5.11 and 5.13, $\Phi\colon \mathrm{Sym}(\Pi) \to \mathrm{Sym}(S^3, Th(p,q))$ is an isomorphism, and by Lemmas 4.2 and 4.3, $\mathrm{Sym}(\Pi) \cong D_{2q}$ of order $4q$. Corollary 1.2 follows by Remark 4.4. □

Remark 6.1. What the proof cites without re-proving: the Main Theorem of [MT93] with their definitions of flat map and flype [MT93, pp. 115–116]; Borel's theorem, via [CWY]; [Mur87, Theorems 2 and 3]; Gordon–Luecke [GL89], only for passing from the pair to the exterior; and standard facts: hyperbolic knot exteriors are aspherical with incompressible boundary and centreless π_1, hyperbolic knots are prime, the peripheral subgroup is malnormal, the centre of a non-trivial amalgam over P lies in P, and the independence of the category noted in Section 5. For general polyhedral diagrams the argument would need an isometric realisation of the diagram automorphisms, e.g. via Koebe–Andreev–Thurston; for $Th(p,q)$ the drawing of Section 5.2 is explicit. Schrijver's flyping theorem for well-connected diagrams [Sch93] is the diagram-level statement matching Lemma 3.1; it gives no statement about maps and is not used.

7. Comparison with Sakuma–Weeks for $p = 3$

What [SW95] *prove.* $K(n)$ is the closure of the 3-braid $(\sigma_1\sigma_2^{-1})^n$ [SW95, Example I.2.2], so $K(q) = Th(3,q)$. For $n \geq 4$, $\mathrm{Isom}(S^3 - K(n)) \cong D_{2n}$, of order $4n$; for $n = 3$, the Borromean rings, the group is $O \times \mathbb{Z}_2$ [SW95, Proposition I.2.5(2)]. $Th(3,q)$ is a knot if and only if $3 \nmid q$, so for knots their range $n \geq 4$ is exactly the range of Theorem 1.1 for $p = 3$. (The survey treats $Th(3,q)$ under the assumption $\gcd(3,q) = 1$ [DS, §5] and states Theorem M "for any q" in that setting; [SW95, Proposition I.2.5] covers $n \geq 4$ there, and $q = 2$, the figure-eight knot, lies outside both their range and ours.)

How they prove it. Following Thurston [Th, 6.8.11], $S^3 - K(n)$ is glued from two ideal polyhedra $P(n)$ whose faces are regular in the Klein model, hyperbolically and euclideanly; this gives the complete hyperbolic structure [SW95, pp. 399–401]. Sakuma and Weeks introduce the construction through the decomposition of an alternating link complement into two polyhedra read off the diagram [SW95, p. 397, Proposition I.2.1], crediting Menasco, Aitchison–Rubinstein and others [Me83, AR1]. A centre criterion [SW95, Proposition I.1.4] shows that this is the canonical (Epstein–Penner [EP]) decomposition. By Mostow rigidity the mapping class group of the complement is the combinatorial automorphism group of the canonical decomposition [SW95, p. 393]; that this group is the one in the proposition is "easy to see" [SW95, p. 402].

How we prove it. The route here is topological and diagrammatic: Theorem 5.6 reduces $\mathrm{Sym}(S^3, K)$ to $\mathrm{Sym}(\Pi)$ once Section 3 has excluded non-trivial flypes, and the $4q$ symmetries are counted in Section 4. No hyperbolic structure is constructed.

Where the two meet. Both arguments end in a finite combinatorial count: in [SW95] the automorphism group of the two polyhedra and their gluing, not written out; here the symmetry group of the 4-valent map with over/under data, written out in Section 4.

What each gives. [SW95] give the canonical decomposition itself, hence the hyperbolic structure explicitly, and the isometry group also for the link $K(3)$. Their first step needs the explicit regular structure, and by their own Remark I.1.5 the canonical decompositions of almost all hyperbolic knots up to 10 crossings do not satisfy the assumption of their criterion. We do not know whether $Th(p,q)$ for $p \geq 4$ has a decomposition to which their Proposition I.1.4 applies; the argument here does not need one. It covers every $p \geq 3$ at once, and it rests instead on the Main Theorem of [MT93], on Borel's theorem, on [Mur87] and, for the exterior, on [GL89].

The relation they point to. In §III.1 Sakuma and Weeks conjecture that the arc of every crossing of a reduced alternating diagram of a hyperbolic alternating link is isotopic to an edge of the canonical decomposition, and write: "We hope this approach sheds new light on the Tait conjecture (which was proved by [MT] using combinatorial methods) from the point of view of hyperbolic geometry" [SW95, p. 436]. For Turk's head knots the argument here runs the other way, from the theorem of Menasco and Thistlethwaite to the symmetry group that Sakuma and Weeks obtained geometrically for $p = 3$. For $p = 3$ the two agree: Section 4 gives ρ, σ and the μ_a, together D_{2q} of order $4q$.

8. Computational checks

The proof does not rest on computation; the following checks test the explicit formulas and the model.

- The formulas of Section 4: ρ, σ and all μ_a are map automorphisms with the stated ε; constant a fails; τ has order $2q$; $\mu\tau\mu = \tau^{-1}$; the $4q$ elements have distinct vertex images. 217 of 217 cases ($3 \leq p \leq 15$, $3 \leq q \leq 30$, $\gcd = 1$).
- The hand model of the diagram is isomorphic, as a map with over/under data, to spherogram's diagram of the braid closure, for even p to its mirror image: 60 of 60 cases ($p \leq 10$, $q \leq 15$).
- Independently of the hand model, on spherogram's PD code: Tait graph simple and 3-connected, $|\operatorname{Sym}(\Pi)| = 4q$, dihedral, in 263 of 263 cases with $q \geq 3$ ($p \leq 16$, at most 200 crossings). Calibration: 8_{18} gives 16, 9_{40} gives 12, 10_{123} gives 20; diagrams with flypes (6_1, 7_6, 8_3, 8_{14}) are correctly not 3-connected.
- The symmetric drawing of Section 5.2 is planar, has four rays at every vertex, and ρ, σ, all μ_a are affine maps of it onto itself: 42 of 42 cases ($p \leq 9$, $q \leq 12$); for $p \geq 4$, with offsets $\frac{i-1}{p-1}$ the maps μ_a are not of the form $\theta \mapsto a - \theta$, that is, not isometries of the drawing. The cyclic order of the four rays: 64 of 64 cases.

- SnapPy's isometry groups (numerical) have order $4q$ in 95 of 95 cases with $q \geq 3$ ($p \leq 11$, at most 90 crossings); exact certification only for $Th(3,4)$ and $Th(4,3)$.

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Tool use. This work was prepared with substantial use of Claude Code (Anthropic), an agentic AI system based on large language models, working together with the author.

- *Mathematics.* Conjecture R was among the open problems found in a literature search carried out with Claude Code; the author chose it. The proof was worked out by Claude Code under the author's direction.
- *Writing and typesetting.* The text and the LaTeX source were written by Claude Code, in English; no translation tools were used. The paper contains no figures.
- *References.* Literature was searched with Claude Code; the sources were obtained by the author and by Claude Code, and W. Menasco sent us [MT93] and [Me83]. Quotations and page references were checked against the original texts where these were available to us; bibliographic data were checked against Crossref or against the sources themselves.
- *Code.* The checks of Section 8 were written and run by Claude Code in Python, using SnapPy 3.3.2, spherogram 2.4.1 and networkx 3.6.1, and SageMath 10.9 for the exact certification. The deposited scripts, at https://doi.org/10.5281/zenodo.23234787, are English versions of the working scripts, prepared by Claude Code and run again before deposit with the same results; the deposit lists the exact software versions.
- *Checking.* Drafts were read at least seventeen times by separate instances of Claude Code (in full, as the changes between versions, or as targeted readings of single passages), without access to the discussion in which the arguments were developed; their findings were checked and incorporated where they held.

The author takes full responsibility for all contents.

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