THE SYMMETRY GROUP OF THE TURK'S HEAD KNOTS $Th(p, 2)$

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Abstract. Di Prisa and Şavk conjectured that the Turk's head knot $Th(p, q)$, $\gcd(p, q) = 1$, has symmetry group D_{2q}, the dihedral group of order $4q$ (Conjecture R). The case $q \geq 3$ is proved in [St]. We treat $q = 2$: for every odd $p \geq 3$, $Th(p, 2)$ is a hyperbolic, amphicheiral two-bridge knot, and hence its symmetry group is the dihedral group of order 8, as conjectured.

1. Statement

For $p \geq 3$ let $\beta_p = \sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\cdots\sigma_{p-1}^{\pm 1} \in B_p$; the Turk's head link $Th(p, q)$ is the closure of β_p^q [DS, §2]. The symmetry group $\mathrm{Sym}(S^3, K)$ of a knot is the mapping class group of its exterior [DS, §5.2]; D_n denotes the dihedral group of order $2n$.

> **Conjecture R** [DS, §6.3]. *Suppose* $\gcd(p, q) = 1$. *Then* $\mathrm{Sym}(S^3, Th(p, q)) \cong D_{2q}$.

The survey assumes $p \geq 3$ and $q \geq 2$ [DS, §4.1]. For $q \geq 3$ the conjecture is proved in [St]; there the case $q = 2$ is excluded because the standard diagram has bigons. For $q = 2$ the condition $\gcd(p, 2) = 1$ means that p is odd.

Theorem 1. *For every odd* $p \geq 3$, $\mathrm{Sym}(S^3, Th(p, 2)) \cong D_4$, *the dihedral group of order* 8. *Hence Conjecture R holds for* $q = 2$.

The proof combines three known facts with one observation (Lemma 2): $Th(p, 2)$ is a two-bridge knot. For $p = 3$ the theorem is known: $Th(3, 2)$ is the figure-eight knot [DS, §4.12], covered by [DS, Theorem M].

2. $Th(p, 2)$ is a two-bridge knot

Let $p \geq 3$ be odd and let D be the closed braid diagram of β_p^2 in S^2, drawn around the braid axis; by definition it is alternating, with $2(p - 1)$

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crossings [DS, §4.2]. Number the $2p-2$ letters of β_p^2 by their angular positions $0, 1, \dots, 2p-3$ around the axis; the letter $\sigma_i^{\pm 1}$ occurs at the positions $i-1$ and $p+i-2$. The diagram runs on p concentric *strand positions* $1, \dots, p$ (counted from the axis). Call the annulus between strand positions i and $i+1$ *column* i ($1 \le i \le p-1$), the disc inside strand position 1 column 0 (the region R_0) and the region outside strand position p column p. For $1 \le i \le p-1$, column i is cut by its two crossings into two regions; let a_i be the one over the open arc of positions $(i-1,\ p+i-2)$ and b_i the other. Comparing positions:

- along its outer side (strand position $i+1$) the region a_i meets exactly one crossing of column $i+1$, at position i (if $i \le p-2$), and along its inner side (strand position i) exactly one crossing of column $i-1$, at position $p+i-3$ (if $i \ge 2$); the region b_i meets the crossings of columns $i+1$ and $i-1$ at the positions $p+i-1$ and $i-2$ (read modulo $2p-2$), under the same conditions. The corners of R_0 are the two crossings of column 1, those of the region of column p the two crossings of column $p-1$.

So for $2 \le i \le p-2$ the corners around a_i (and around b_i) are, in cyclic order: a crossing of column i, one of column $i+1$, the other crossing of column i, one of column $i-1$; for $i = 1$ and $i = p-1$ the corner of the missing neighbouring column is absent and there are three corners. In particular every region meets every crossing at most once, so no crossing has the same region at two opposite corners and D has no nugatory crossing: it is a reduced alternating diagram with $2(p-1)$ crossings. We also record, for strand position $i+1$ with $1 \le i \le p-2$: the crossings on it are those of columns i and $i+1$, at the positions $i-1,\ i,\ p+i-2,\ p+i-1$, so along strand position $i+1$ the crossings of the two columns alternate.

Conway circles. For even i with $2 \le i \le p-1$ let C_i be a simple closed curve that runs once around the axis inside column $i-1$, except near the two crossings of column $i-1$ (positions $i-2$ and $p+i-3$), which it passes on their outer side, through the regions b_i and a_i. Near each such crossing C_i crosses the two strands of the crossing once each, both times on strand position i, and nowhere else does it meet D; so C_i meets D in four points. Let B_i be the disc bounded by C_i on the side of column p, and $T_i = D \cap B_i$. It is a tangle, and its crossings are exactly those of columns $i, i+1, \dots, p-1$.

We use the twist form of rational tangles [KL, Definition 1]: a tangle obtained from $[0]$ or $[\infty]$ by consecutive additions $T + [\pm 1]$, $[\pm 1] + T$ and multiplications $T * [\pm 1]$, $[\pm 1] * T$ is rational. Geometrically, each operation adds one crossing between two neighbouring endpoints of T outside its disc; whether a given crossing is a sum or a product depends only on how the disc is framed, and rationality does not depend on that choice.

Lemma 2. *For odd $p \ge 3$, $Th(p, 2)$ is a two-bridge knot.*

Proof. T_{p-1} consists of the two crossings of column $p-1$ (and arcs of strand positions $p-1$ and p). Each of them joins two neighbouring endpoints of

the tangle formed by the other one, which is $[\pm 1]$; so T_{p-1} is obtained from $[\pm 1]$ by one operation of [KL, Definition 1] and is rational.

Let $i \leq p-3$ be even. The disc B_i contains B_{i+2} and four further crossings: the two of column $i+1$ and the two of column i. The curve C_{i+2} passes each crossing x of column $i+1$ on its outer side, and the two points where C_{i+2} meets D near x lie on the two strands of x; so x joins two neighbouring endpoints of T_{i+2}, and the two crossings of column $i+1$ do so at the two different pairs of neighbouring endpoints (they lie at the two different places where C_{i+2} leaves column $i+1$). Adding them is a sum on either side: $T' = [\pm 1] + T_{i+2} + [\pm 1]$. Its disc may be taken to be B_{i+2} enlarged by small neighbourhoods of the two crossings; it is bounded by a curve that runs in column $i+1$ and passes the two crossings of column $i+1$ (positions i and $p+i-1$) on their inner side, through a_i and b_i. This curve meets D in four points, all on strand position $i+1$: one on each of the four arcs into which the four crossings at the positions $i-1$, i, $p+i-2$, $p+i-1$ cut strand position $i+1$, and in this cyclic order. The two points on the arcs at position i are the endpoints of T' coming from x (position i), the two on the arcs at position $p+i-1$ those coming from the other crossing x'. Since the crossings of columns i and $i+1$ alternate along strand position $i+1$, the column-i crossing at position $i-1$ lies on the arcs $(p+i-1,\, i-1)$ and $(i-1,\, i)$ and so joins one endpoint coming from x' to the neighbouring one coming from x; likewise the crossing at position $p+i-2$ joins the other two. Between the enlarged disc and C_i there are no further crossings. So the two crossings of column i join neighbouring endpoints of T' at the two pairs other than those used by the sums, and $T_i = [\pm 1] * ([\pm 1] + T_{i+2} + [\pm 1]) * [\pm 1]$ (one product on each side). By induction T_2 is rational.

Outside B_2 lie the two crossings of column 1 and the disc R_0. The curve C_2 passes both crossings of column 1 on their outer side, so as above they add $[\pm 1] + T_2 + [\pm 1]$, a rational tangle T_0; outside its disc there are no crossings, only two arcs that join its four endpoints in pairs. So D is the numerator or the denominator closure of the rational tangle T_0, and $Th(p,2)$ is a rational knot, i.e. a two-bridge knot [KLk, §1]. □

Remark 3. Numerically, for every odd $p \leq 21$, $Th(p,2)$ is the two-bridge knot $C(2,2,\ldots,2)$ with $p-1$ entries 2 (fractions $5/2$, $29/12$, $169/70, \ldots$; determinant the Pell number, in agreement with [DS, Theorem H]); see Section 4. Neither the proof of Lemma 2 nor that of Theorem 1 uses this.

3. Proof of Theorem 1

Let $p \geq 3$ be odd and $K = Th(p,2)$, with complement $M = S^3 \setminus K$.

K is hyperbolic. By Lemma 2 it is a two-bridge knot. Apart from the trivial link of one or two components, every two-bridge link is a link $K(\Omega)$ of Futer's construction, and such a link that is not hyperbolic has a diagram with a single twist region and is a torus link [Gu, Appendix, the paragraph before Theorem A.1, Theorem A.1 and the paragraph after it]; a diagram

with a single twist region shows a torus link $T(2,n)$. K has a reduced alternating diagram with $2(p-1) \geq 4$ crossings, so its crossing number is $2(p-1)$ [Kau, Mu, Thi]. The knots among the $T(2,n)$ with crossing number at least 3 have n odd and crossing number $|n|$, which is odd; $2(p-1)$ is even. So K is not $T(2,n)$, and it is hyperbolic.

K is amphicheiral. Krötenheerdt's rosette knot R_n^m is the closure of $(\sigma_1\sigma_2^{-1}\cdots\sigma_{n-2}\sigma_{n-1}^{-1})^m$ for odd n, with $m,n>1$ coprime [Kr, pp. 270–271], that is, $Th(n,m)$ (up to the sign convention for the braid generators, which does not matter for amphicheirality); by [Kr, Satz 1, p. 272] every R_n^m of odd order n is isotopic to its mirror image. (The survey quotes this as positive and negative amphicheirality [DS, §4.4]; we use only the unoriented statement.) For completeness, here is a direct argument. Let β_p^* be β_p with all exponents negated, $\mathrm{rev}(w)$ the word w read backwards, and ϕ the substitution $\sigma_i \mapsto \sigma_{p-i}$. Since p is odd, $\beta_p^* = \mathrm{rev}(\phi(\beta_p))$, hence $(\beta_p^*)^2 = \mathrm{rev}(\phi(\beta_p^2))$. The map $z \mapsto 1/z$ of S^2 is orientation preserving and carries the closed braid diagram of a word w around the axis point 0 to that of $\mathrm{rev}(\phi(w))$ (it exchanges strand positions i and $p+1-i$ and reverses the angular direction, keeping over- and undercrossings); so the closures of w and $\mathrm{rev}(\phi(w))$ are the same link. The closure of $(\beta_p^*)^2$ has the diagram D with all crossings switched, i.e. it is the mirror image of K. Hence K is isotopic to its mirror image. So there is an orientation-reversing homeomorphism of S^3 carrying K to itself; its restriction is an orientation-reversing self-homeomorphism of M.

The group. Write $K = K(\Omega)$ in the notation of Millichap and Worden (every two-bridge link can be written so [MW, §2]), and let $\mathrm{Sym}(M)$ be the group of self-homeomorphisms of M up to isotopy and $\mathrm{Sym}^+(M)$ its orientation-preserving subgroup. By [MW, Theorem 4.1] (which combines Sakuma–Weeks [SW, Theorem II.3.2, Lemma II.3.3] with the canonicity of the Sakuma–Weeks triangulation proved by Guéritaud [GuT]), for a hyperbolic two-bridge link complement $\mathrm{Sym}(M) = \mathrm{Sym}^+(M)$ unless Ω is palindromic and its number n of syllables [MW, §2] is even, and in that case $\mathrm{Sym}(M) \cong D_4$ and $\mathrm{Sym}^+(M) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since M has an orientation-reversing symmetry, $\mathrm{Sym}(M) \neq \mathrm{Sym}^+(M)$, so $\mathrm{Sym}(M) \cong D_4$. By [MW, Proposition 4.2], $\mathrm{Sym}(M)$ is isomorphic to the group of self-homeomorphisms of the pair (S^3,K) up to isotopy (this is what $\mathrm{Sym}(S^3,K)$ denotes in [MW, §4]). For a non-trivial knot, the latter group is naturally isomorphic to the mapping class group of the knot exterior [Ko, §1.3 and Proposition 1.4], a consequence of the theorem of Gordon and Luecke ([Ko] works in the piecewise linear category, [MW] with homeomorphisms; we rely on the classical fact that for tame knots in S^3 the classification of homeomorphisms of pairs up to pairwise isotopy, and of homeomorphisms of the exterior up to isotopy, does not depend on the choice between the topological, piecewise linear and smooth categories); and the mapping class group of the exterior is the group $\mathrm{Sym}(S^3,K)$ of [DS, §5.2]. Hence $\mathrm{Sym}(S^3,K) \cong D_4$, of order $8 = 4q$. □

4. Checks

Computer checks with SnapPy 3.3.2 inside SageMath 10.9 (SnapPy's determinant needs Sage) and NetworkX, independent of the proof. For odd $p \le 41$: both checkerboard graphs of the closed braid diagram of β_p^2 are isomorphic to the graph with vertices R_0 and a_i, b_i (i even), with two parallel edges a_ib_i for each even i, the edges R_0a_2, R_0b_2, and the edges a_ib_{i+2}, b_ia_{i+2} for even $i \le p-3$ (one edge for each crossing, read off from the corners in Section 2), and in both, around every vertex of degree 4 (these are the regions a_i, b_i with $2 \le i \le p-2$: even i in one checkerboard graph, odd i in the other, $p-3$ in each) the two crossings of its own column are not adjacent in the cyclic order (the order is: own column, other, own column, other), as used for the Conway circles. For odd $p \le 21$: the determinant of $Th(p,2)$ is the Pell number P_p, SnapPy's (not interval-verified) isometry test identifies $Th(p,2)$ with $C(2,\dots,2)$ ($p-1$ entries), and SnapPy's symmetry group is D_4 (full group, amphicheiral). For odd $p \le 15$ and $q = 2, 3$: the word identity $\beta_p^* = \mathrm{rev}(\phi(\beta_p))$ holds; for the knots among these, the signature is 0, the Jones polynomial satisfies $V(t) = V(t^{-1})$, and SnapPy (not interval-verified) finds self-isometries of the exterior whose action on the cusp torus has determinant -1, i.e. orientation-reversing ones. Scripts and outputs: **[deposit to be inserted]**.

Tool use. This work was prepared with substantial use of Claude Code (Anthropic), an agentic AI system based on large language models, working together with the author.

- *Mathematics.* The case $q = 2$ was left open in [St]; the author chose to treat it. The two-bridge lemma and the proof were worked out by Claude Code under the author's direction.
- *Writing and typesetting.* The text and the LaTeX source were written by Claude Code, in English. The note contains no figures.
- *References.* Literature was searched with Claude Code. [Kr] was read in the GDZ digitization, the passages used transcribed from the page images; statements from [DS], [Gu], [GuT], [MW], [SW], [Ko], [KL] and [KLk] were checked against the texts named in the bibliography; [Kau], [Mu] and [Thi] are cited for the crossing number of reduced alternating diagrams as in [St].
- *Code.* The checks of Section 4 were written and run by Claude Code in Python, with SnapPy 3.3.2 and spherogram inside SageMath 10.9 and with NetworkX; the scripts and their outputs are deposited with this note.
- *Checking.* Drafts were read by separate instances of Claude Code without access to the discussion in which the arguments were developed: three times in full on 17 October 2026 and twice as diff reads on 18 October 2026, and once in full by a separate Claude session on 18 October 2026; their findings were checked and incorporated where they held.

The author takes full responsibility for all contents.

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