PRIME KNOTS WITH TWISTED ALEXANDER VANISHING ORDER 24 UP TO 19 CROSSINGS

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Abstract. Ishikawa, Morifuji and Suzuki defined the twisted Alexander vanishing order $O(K)$ of a knot, showed $O(K) \geq 24$, with equality exactly when the symmetric group $\mathfrak{S}_4$ is a twisted Alexander vanishing group of K; among the prime knots with at most ten crossings, $O(K) = 24$ holds exactly for 9_{35} and 9_{46}, as shown by Morifuji and Suzuki. We decide $O(K) = 24$ for each of the 352,152,252 prime knots with at most 19 crossings in Burton's census: it holds for exactly 18,436 of them. There are none with 10 or 11 crossings; with 12 crossings they are $12a_{270}$, $12n_{333}$, $12n_{809}$ and $12n_{868}$.

1. Introduction

Let K be a knot with group $G(K)$. Following Ishikawa, Morifuji and Suzuki, a finite group G is a *twisted Alexander vanishing (TAV) group* of K if some epimorphism $f\colon G(K) \to G$ has twisted Alexander polynomial zero for the composition of f with the regular representation of G, and the TAV order $O(K)$ is the smallest order of such a group ($O(K) = \infty$ if there is none) [IMS1]. They prove $O(K) \geq 24$ for every knot [IMS1, Theorem 1.2(i)], and $\mathfrak{S}_4$ is the only TAV group of order 24 [IMS3]. Among the prime knots with at most ten crossings, $O(K) = 24$ holds exactly for 9_{35} and 9_{46} [IMS1, Theorem 1.3], shown first by Morifuji and Suzuki [MS] (see [IMS1, §3.2]). In a sequel they write [IMS2, p. 3]: "At this point, we do not know how the TAV order $O(K)$ behaves when we restrict ourselves to special classes of knots, e.g., knots with a given bridge number, genus, braid index, crossing number, and so on." The authors now call that paper an unpublished manuscript [IMS3, §1]; we cite it only for the question, for its Theorem 1.7 on two-bridge knots and for its attribution of $O(9_{35}) = O(9_{46}) = 24$ to [MS] ([IMS2, after Theorem 1.5]).

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This work was developed in collaboration with the AI system Claude Code (Anthropic): the choice of problem, the direction of the work and the computing hardware are due to the author (a sculptor); the formalisation, the algorithms and the proofs were developed by Claude Code (Anthropic) working with the author. The collaboration is stated as a fact, not a disclaimer; details are in the section on tool use at the end.

Theorem 1. *Among the* 352,152,252 *prime knots with at most* 19 *crossings in Burton's census* [Bur]*, exactly* 18,436 *have* $O(K) = 24$*. Their numbers by crossing number and type are given in Table 1; none is fibred.*

c	alternating	non-alternating hyperbolic	satellite
≤ 8	0	0	0
9	1	1	0
10, 11	0	0	0
12	1	3	0
13	1	4	0
14	1	19	0
15	11	63	2
16	28	244	2
17	50	737	2
18	201	2,947	13
19	696	13,378	31
total	990	17,396	50

TABLE 1. Prime knots with $O(K) = 24$. No torus knot occurs. The alternating ones are hyperbolic. Tested: 51,280,967 alternating hyperbolic, 300,870,891 non-alternating hyperbolic, 380 satellite and 14 torus knots.

The knots with 12 crossings are $12a_{270}$, $12n_{333}$, $12n_{809}$, $12n_{868}$ (Burton's 12ah_0152, 12nh_149, 12nh_143, 12nh_388; Burton's table names and SnapPy's identification agree), and those with 13 crossings are $13a_{3121}$, $13n_{2400}$, $13n_{3663}$, $13n_{3665}$, $13n_{4384}$ (SnapPy's identification of Burton's diagrams). A fibred knot has monic twisted Alexander polynomials [Cha], so $O(K) = \infty$ by definition [IMS1, §1]; that none of the 18,436 is fibred is therefore a consistency check, not an additional result.

2. Method

Reduction. A twisted Alexander polynomial of a direct sum of representations is the product of the factors, and the regular representation of $\mathfrak{S}_4$ contains the trivial, sign, two-dimensional and both three-dimensional irreducible representations. For an epimorphism $f\colon G(K) \to \mathfrak{S}_4$ the meridians go to one conjugacy class of odd permutations (the abelianisation of $\mathfrak{S}_4$ is $\mathbb{Z}/2$), so they are transpositions or 4-cycles. The one-dimensional factors give $\Delta_K(\pm t) \neq 0$; the two-dimensional representation factors through $\mathfrak{S}_3$, which is not a TAV group since $|\mathfrak{S}_3| < 24$; and the three-dimensional representation twisted by the sign is the standard one evaluated at $-t$. So the regular polynomial vanishes if and only if the polynomial of the standard representation vanishes, that is, if and only if the polynomial of the four-dimensional

permutation representation (standard plus trivial, the latter contributing $\Delta_K(t)$) vanishes.

Enumeration. From a Wirtinger presentation read off the census diagram we enumerate all homomorphisms $G(K) \to \mathfrak{S}_4$ sending the meridians to transpositions or to 4-cycles, keep the surjective ones, one per conjugacy class, and compute the Wada numerator of the permutation representation. It is a polynomial of degree at most $N = 4(c-1)$ in t. The polynomial of [IMS1] is the order of the twisted homology module H_1 [IMS1, §2]. Wada's invariant, the Wada numerator divided by $\det(\rho(x)t - I)$ for a meridian x, equals the order of H_1 divided by the order of H_0, up to units [FV, §9.1, Remark (4)]. Both divisors are non-zero: $\det(\rho(x)t - I)$ is $\pm(t-1)^3(t+1)$ for a transposition and $\pm(t^4-1)$ for a 4-cycle, and H_0 is a quotient of the module presented by this matrix. So the polynomial of [IMS1] vanishes if and only if the Wada numerator does.

Decision. A value that is non-zero modulo $p = 2^{31} - 1$ at some integer t shows that the polynomial is not zero. For an epimorphism whose value is zero at two pseudo-random points (fixed seed) we evaluate the integer determinant at the $N+1$ integers $t = 2, \ldots, N+2$, modulo enough primes that their product exceeds twice the Hadamard bound; if all these values are zero, the polynomial is identically zero. The census run used a GPU for both steps.

Independent checks.

- A CPU reference program with its own enumeration and determinant code, run on KnotInfo diagrams, gives exactly $\{9_{35}, 9_{46}\}$ among the 249 prime knots with at most ten crossings, as in [IMS1, Theorem 1.3]; the census run gives the same two knots (Table 1; identified as 9_{35} and 9_{46} by SnapPy, see the deposit). The reference program evaluates both the regular and the four-dimensional permutation representation: on all 860 epimorphisms of these 249 knots that send the first meridian to a fixed transposition or 4-cycle (every epimorphism is conjugate to one of them) the two agree on vanishing at two pseudo-random points modulo p, a check of the reduction above on this range.
- For every one of the 18,436 knots, one stored vanishing epimorphism was checked again on the CPU without GPU arithmetic: the Wirtinger relations, surjectivity, the meridian class, and the vanishing of the exact integer determinant at all $N+1$ points (SageMath [Sage]). All 18,436 certificates passed; there were no failures (run of 16 October 2026, 1,459 input files).
- A second script built on the CPU reference program of the first item confirmed 30 randomly chosen knots with $O(K) = 24$ and 30 randomly chosen knots without.
- The same CPU route decided every one of the 313,230 census knots with at most 15 crossings from scratch, without using the GPU result

in the decision (the list of knots is taken from the block sizes of the GPU output; runs of 19 to 21 October 2026, about 940 worker-hours): an epimorphism whose determinant is non-zero modulo p at one pseudo-random integer t counts as non-vanishing; otherwise the exact integer determinant is evaluated at $t = 2, \dots, N+2$ until a value is non-zero, and a knot is a hit at the first epimorphism for which all $N+1$ values vanish. It finds the same 107 knots with $O(K) = 24$ as the census run, knot by knot, and for none of the 313,123 other knots was an epimorphism zero modulo p without vanishing exactly. It reads the same Wirtinger presentations as the census run and so does not check their conversion.

- Consistency: none of the 990 alternating knots with $O(K) = 24$ has monic Alexander polynomial, so none is fibred [Mur]; all 4,010 non-alternating ones whose knot Floer homology was computed (15–18 crossings, with the function `knot_floer_homology` of SnapPy) are non-fibred, by Ni's theorem [Ni]; none of the eleven knots with $O(K) = 24$ and at most 13 crossings is a two-bridge knot (KnotInfo lists 714 two-bridge knots up to 13 crossings), in line with [IMS2, Theorem 1.7].

Remark 2. The theorem is computer-assisted. That a listed knot has $O(K) = 24$ is certified for each knot by exact integer arithmetic. That no other knot of the census has $O(K) = 24$ rests on the completeness of the enumeration of epimorphisms in the GPU program, on its modular arithmetic (a spurious non-zero value would hide a vanishing polynomial), on the conversion of the census diagrams to Wirtinger presentations, and on Burton's census. The GPU program was compared with the CPU reference code, which is independent of it, by outcome: knot by knot on the random sample above, by the set of knots with $O(K) = 24$ on all knots with at most ten crossings, and knot by knot on all 313,230 census knots with at most 15 crossings. Up to 15 crossings the absence of further knots therefore rests on two programs with independent code for the enumeration and the arithmetic; they share Burton's census, the Wirtinger presentations and their convention, the list of knots (read from the GPU output; its length, 313,230, is the number of census knots with at most 15 crossings), the reduction to the permutation representation and the degree bound N. Composite knots were not considered.

Tool use. This work was prepared with substantial use of Claude Code (Anthropic), an agentic AI system based on large language models, working together with the author.

- *Mathematics.* The question of [IMS2] was found in a literature search carried out with Claude Code and chosen by the author. The reduction, the census computation and the certificates were worked out by Claude Code under the author's direction.
- *Writing and typesetting.* The text and the LaTeX source were written by Claude Code, in English; no translation tools were used. The note contains one table and no figures.

- *References.* Literature was searched with Claude Code. Quotations and theorem numbers were checked against the arXiv versions named in the bibliography; the attribution to [MS] follows [IMS1] and [IMS2]; bibliographic data were checked against Crossref or DataCite or against the sources themselves.
- *Code.* The computations were written and run by Claude Code in Python, using CuPy on an NVIDIA GPU, Regina, SnapPy 3.3.2 (with `knot_floer_homology` 1.2.2 for knot Floer homology) and SageMath 10.9 for the exact CPU certificates; the deposit lists the software versions.
- *Checking.* Drafts were read by separate instances of Claude Code, without access to the discussion in which the arguments were developed; their findings were checked and incorporated where they held.

The author takes full responsibility for all contents.

Data

The list of the 18,436 knots with their vanishing epimorphisms, the per-crossing-number counts, the programs and the logs are deposited at **[Zenodo DOI to be inserted]**.

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