# POSITIVE AMPHICHIRALITY OF TURK'S HEAD KNOTS: THE ORDERS OF THE MAPS

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ABSTRACT. Let $p \geq 3$ be odd, $q \geq 3$ and $\gcd(p, q) = 1$. We show that the Turk's head knot $Th(p, q)$ is strongly positive amphichiral if and only if $q$ is odd, and that the orders of the finite-order diffeomorphisms realising its positive amphichirality are exactly the numbers $2q/d$, $d$ an odd divisor of $q$. This answers Problem D of Di Prisa and Şavk for these knots. The proof reads the answer off the symmetry group $D_{2q}$ determined in a companion paper, together with the action of each symmetry on the orientation of the knot. For $q = 2$ and $p \geq 5$ odd we show that the two positive amphichiral classes in the symmetry group $D_4$ of $Th(p, 2)$ have order 4; hence every finite-order homeomorphism realising the positive amphichirality of $Th(p, 2)$ has order divisible by 4, and $Th(p, 2)$ is not strongly positive amphichiral even with topological involutions. Moreover, every finite-order diffeomorphism realising it has order exactly 4; we do not show that such a diffeomorphism exists. For $p = 3$, the figure-eight knot, only the piecewise linear non-strong-positivity of Hartley and Kawauchi is available here, and the orders remain conditional.

## 1. RESULT

The Turk's head link $Th(p, q)$ is the closure of the $p$-strand braid $\beta_p^q$, $\beta_p = \sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\cdots\sigma_{p-1}^{\pm 1}$; it is a knot if and only if $\gcd(p, q) = 1$. Following [DS, §4.4], a knot $K \subset S^3$ is *positive amphichiral* if there is an orientation-reversing homeomorphism of $S^3$ carrying $K$ to itself with its orientation preserved; *periodically* positive amphichiral if such a map can be chosen of finite order, and *strongly* positive amphichiral if it can be chosen to be an involution. For $p$ odd, $Th(p, q)$ is periodically positive amphichiral [DS, §4.4, Theorem D]; and it is strongly positive amphichiral if $p$ and $q$ are both odd, as observed by Long [DS, §4.4]. Di Prisa and Şavk ask:

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**Problem D** [DS, §6.3]. Let $p$ be odd. Determine which Turk's head links $Th(p,q)$ are strongly positive amphichiral. Moreover, determine the orders of the maps realizing periodic positive amphichirality.

**Theorem 1.** *Let $p \geq 3$ be odd, $q \geq 3$ and $\gcd(p,q) = 1$, and let $K = Th(p,q)$.*

(i) *A finite-order diffeomorphism of $S^3$ realising the positive amphichirality of a smooth representative of $K$ has order $2q/d$ for some odd divisor $d$ of $q$. Every such order is realised, by an isometry of $S^3$ of that order preserving a smooth representative of $K$. Every finite-order homeomorphism realising the positive amphichirality has order a multiple of some $2q/d$.*

(ii) *$K$ is strongly positive amphichiral, with the involution allowed to be any homeomorphism, if and only if $q$ is odd.*

(iii) *Every orientation-reversing symmetry of $(S^3, K)$ that reverses the orientation of $K$ is isotopic to an involution, and there are exactly $q$ such symmetry classes, as there are $q$ positive amphichiral ones.*

The theorem covers knots only, and $q \geq 3$. The case $q = 2$ is treated, in part, in Section 4 (Theorem 7); the links ($\gcd(p,q) > 1$) of Problem D are not treated here. Part (ii) is known with the involution piecewise linear: for $q$ odd it is Long's observation, and for $q$ even it follows from the theorem of Hartley and Kawauchi that the Alexander polynomial of a strongly positive amphichiral knot is a square [HK, Theorem (2)], where strong amphichirality is defined with piecewise linear involutions [HK, p. 63], together with Saurabh's theorem that $\Delta_{Th(p,q)}$ is not a Laurent square for $p$ odd and $q$ even [Sau, Theorem 1.5]. For $q \geq 3$ our argument is independent of both, allows any homeomorphism as the involution in (ii), and gives in addition the orders in (i), for which we have not found a reference. Examples: $Th(3,4) = 8_{18}$ is positive amphichiral, with finite-order diffeomorphisms of order 8 only, and not strongly so; $Th(3,5) = 10_{123}$ is strongly positive amphichiral, with finite-order diffeomorphisms of orders 2 and 10. Both agree with Lamm's census [Lam, §5], where $8_{18}$ is fully amphichiral and $10_{123}$ is one of the two strongly positive amphichiral knots with 10 crossings.

## 2. The symmetry group, with the orientation of the knot

We use the companion paper [St], whose notation we follow. There $Th(p,q)$ for $p, q \geq 3$, $\gcd(p,q) = 1$, is realised as a curve $\lambda(\Pi) \subset S^3$ over a drawing $\Pi$ on the projection sphere $S^2$ in cylinder coordinates $(\theta, h)$, $\theta \in \mathbb{R}/q\mathbb{Z}$, and the following is proved.

(A) [St, Sections 4 and 5] The group $G$ generated by the orthogonal maps $\rho\colon \theta \mapsto \theta + 1$, $\mu_a\colon \theta \mapsto a - \theta$ ($a \in \mathbb{Z}_q$, $h$ fixed) and $\sigma\colon \theta \mapsto \frac{p-2}{2} - \theta$, $h \mapsto p - h$, each combined with the sign $\varepsilon$ of the height coordinate given in [St, Section 4], consists of the $4q$ isometries $\rho^k, \mu_a, \sigma\rho^k, \sigma\mu_a$

of $(S^3, \lambda(\Pi))$, and the map $\Phi\colon G \to \mathrm{Sym}(S^3, K)$ to isotopy classes of homeomorphisms of the pair is an isomorphism [St, Section 6].

(B) [St, Lemma 4.2 and Remark 4.4] $(g, \varepsilon)$ preserves the orientation of $S^3$ if and only if $\mathrm{sign}(g) \cdot \varepsilon = +1$; here sign and $\varepsilon$ are $(+, +1)$ for $\rho$, $(+, (-1)^p)$ for $\sigma$ and $(-, -1)$ for $\mu_a$.

(C) [St, proof of Lemma 4.3] $\tau_a := \sigma\mu_a$ satisfies $\tau_a^2 = \rho^{p-2-2a}$, and $\sigma$ exchanges the two faces $F_0, F_p$ of degree $q$ while $\rho$ and $\mu_a$ fix them.

**Lemma 2.** *Let $p$ be odd. The elements of $G$ that reverse the orientation of $S^3$ are the $2q$ elements $\sigma\rho^k$ and $\sigma\mu_a$. The $\sigma\rho^k$ reverse the orientation of $K$ and are involutions. The $\tau_a = \sigma\mu_a$ preserve the orientation of $K$, and $\tau_a$ has order $2q/\gcd(p-2-2a,\, q)$.*

*Proof.* By (B) with $p$ odd, $\rho$ and $\mu_a$ preserve the orientation of $S^3$ and $\sigma$ reverses it, so the orientation-reversing elements are $\sigma\rho^k$ and $\sigma\mu_a$.

Orient $K$ as the closure of the braid. In the drawing every edge of $\Pi$ is then traversed in the direction of increasing $\theta$, and an element $(g, \varepsilon)$ acts on the projection of $\lambda(\Pi)$ by $g$. So it preserves the orientation of $K$ exactly when $g$ preserves the direction of $\theta$: $\rho$ does, $\mu_a$ and $\sigma$ reverse it. Hence $\sigma\rho^k$ reverses the orientation of $K$ and $\sigma\mu_a$ preserves it.

The maps are given in coordinates, so the orders can be computed directly. $\sigma^2 = 1$ and $\sigma\rho\sigma = \rho^{-1}$ (both $\theta \mapsto \theta - 1$ after the substitution), so $(\sigma\rho^k)^2 = \sigma\rho^k\sigma\rho^k = 1$, and $\sigma\rho^k \neq 1$. For $\tau_a$: by (C) the odd powers of $\tau_a$ exchange $F_0$ and $F_p$, so they are not the identity, and $\tau_a^{2m} = \rho^{m(p-2-2a)}$. Since $\rho$ has order $q$, $\rho^{p-2-2a}$ has order $q/\gcd(p-2-2a, q)$, and the order of $\tau_a$ is twice that. □

**Lemma 3.** *Let $p$ be odd, $q \geq 3$, $\gcd(p, q) = 1$. As $a$ runs through $\mathbb{Z}_q$, the numbers $d_a = \gcd(p-2-2a,\, q)$ run through exactly the odd divisors of $q$.*

*Proof.* If $q$ is odd, 2 is invertible mod $q$, so $p-2-2a$ runs through all residues mod $q$, and $\gcd(u, q)$ for $u \in \mathbb{Z}_q$ runs through all divisors of $q$, which are odd. If $q$ is even, $p$ is odd, so $p-2-2a$ runs through exactly the odd residues mod $q$; their gcd with $q$ is odd, and every odd divisor $d$ of $q$ occurs, as $\gcd(d, q) = d$ for the odd residue $d$. □

## 3. From classes to maps

The question asks for orders of maps, while $\mathrm{Sym}(S^3, K)$ consists of isotopy classes. For $q \geq 3$ the two agree (Lemma 5). Here maps are diffeomorphisms and $K$ is a smooth knot; isotopy classes are taken as in [St, Section 5], which relies on the classical fact that the choice of category does not affect them.

The proof of injectivity in [St, Section 5.3] uses, as standard facts without a reference, that the double $DE$ of the knot exterior is closed and aspherical, that $\pi_1(DE)$ has trivial centre, and that the peripheral subgroup is malnormal. We give references for them in Lemma 4; it is used again in Remark 10. A knot is hyperbolic if its complement has a complete hyperbolic metric of finite volume [DS, §4.10]; $Th(p, q)$ is hyperbolic [DS, §4.10, Theorem I].

**Lemma 4.** *Let $K \subset S^3$ be a hyperbolic knot, $E$ its exterior and $DE = E \cup_{\partial E} E$ the double of $E$.*

(a) *$DE$ is a closed, orientable, irreducible, aspherical 3-manifold.*
(b) *The centre of $\pi_1(DE)$ is trivial.*
(c) *The peripheral subgroup $\pi_1(\partial E)$ is malnormal in $\pi_1(E)$.*

*Proof.* $E$ is compact and orientable with torus boundary, and its interior, homeomorphic to $S^3 \setminus K$, has a complete hyperbolic metric of finite volume. So $E$ is irreducible and boundary irreducible [Pu, Theorem 8.17], i.e. contains no essential sphere and no essential disk [Pu, Definitions 8.11 and 8.12]; in particular $E$ is not a solid torus, whose meridian disk is essential. Hence $\partial E$ is incompressible in $E$ [AFW, (C.17)], which in [AFW, §1.3] means that $\pi_1(\partial E) \to \pi_1(E)$ is injective.

(a) Let $T \subset DE$ be the common boundary torus of the two copies of $E$. Splitting $DE$ along $T$ gives the two copies of $E$, and both inclusions of $T$ into them are injective on $\pi_1$; so $T$ is incompressible in $DE$ [Ha, §1.2, (1)], and since $E$ is irreducible, $DE$ is irreducible [Ha, §1.2, (4)]. $DE$ is closed, and orientable as the double of an orientable manifold. $\pi_1(E)$ is a retract of $\pi_1(DE)$ [AFW, Lemma 1.6 and its proof], and it contains $\pi_1(\partial E) \cong \mathbb{Z}^2$; so $\pi_1(DE)$ is infinite, and $DE$ is aspherical [AFW, (C.1)], [Ha, Corollary 3.9(a)].

(b) Hyperbolic 3-manifolds (of finite volume) are atoroidal [AFW, §1.5, before Theorem 1.11], so the empty collection of tori satisfies the JSJ Decomposition Theorem [AFW, Theorem 1.7] for $E$: $E$ has no JSJ tori and is its own JSJ component, which is hyperbolic. By [AFW, Theorem 1.16], $T$ is therefore a JSJ torus of $DE$. If $DE$ were Seifert fibred, the empty collection would satisfy [AFW, Theorem 1.7] for $DE$, and the JSJ tori of $DE$, a collection with the minimal number of components, would be empty. So $DE$ is not Seifert fibred. $\pi_1(DE)$ is torsion-free [AFW, (C.2)]. If $z \neq 1$ were central in $\pi_1(DE)$, then $\langle z \rangle$ would be a normal infinite cyclic subgroup, and $DE$ would be Seifert fibred [AFW, Theorem 3.5]. So the centre is trivial.

(c) This is [AFW, Theorem 3.8(1)], applied to the boundary torus $\partial E$, whose JSJ component, $E$ itself, is hyperbolic. □

**Lemma 5.** *Let $f$ be a diffeomorphism of $(S^3, K)$ of finite order $n$. Then the class of $f$ in $\mathrm{Sym}(S^3, K)$ has order $n$.*

*Proof.* Let $m$ be the order of the class, so $m \mid n$, and suppose $m < n$. Then $h = f^m$ is a non-trivial map of finite order, isotopic to the identity as a map of pairs. The finite cyclic group $\langle h \rangle$ acts on $(S^3, K)$ by diffeomorphisms. Averaging a Riemannian metric over $\langle h \rangle$ makes it act by isometries; then a small metric tubular neighbourhood of $K$ is invariant, and $\langle h \rangle$ acts on the exterior $E$. The action is effective, since an isometry of the connected manifold $S^3$ that fixes the open set $\operatorname{int} E$ pointwise is the identity; so $\langle h \rangle$ acts effectively, by the same map on both copies, on the double $DE$. The proof of injectivity in [St, Section 5.3 and Proposition 5.4] uses of the symmetry only that it is a finite-order map preserving $E$ and pairwise isotopic to the

identity, so it applies to $h$: using the malnormality of the peripheral subgroup (Lemma 4 (c)) it shows that the induced outer automorphism of $\pi_1(DE)$ is trivial, and Borel's theorem [Bor], [CWY, pp. 1–2], which applies since $DE$ is closed and aspherical with centreless fundamental group (Lemma 4 (a),(b)), then forces the action to be trivial. So $h = \mathrm{id}$, a contradiction. □

*Proof of Theorem 1.* (i) Let $f$ be a finite-order homeomorphism realising the positive amphichirality of a representative $K'$ of $K$, of order $n$. A homeomorphism of pairs $(S^3, K') \to (S^3, \lambda(\Pi))$ conjugates $\mathrm{Sym}(S^3, K')$ to $\mathrm{Sym}(S^3, \lambda(\Pi))$, preserving orders and the action on both orientations; we identify the two. The class of $f$ reverses the orientation of $S^3$ and preserves that of $K$; both properties are invariant under isotopy of pairs. By (A) the class is $\Phi(g)$ for a unique $g \in G$, and by Lemma 2 $g = \tau_a$ for some $a$. Since $\Phi$ is an isomorphism, the class has the order of $\tau_a$, which is $2q/d_a$ (Lemma 2), with $d_a$ an odd divisor of $q$ (Lemma 3); and the order of the class divides $n$. If $f$ is a diffeomorphism and $K'$ is smooth, Lemma 5 gives $n = 2q/d_a$. Conversely, for every odd divisor $d$ of $q$, Lemma 3 gives $a$ with $d_a = d$, and the isometry $\tau_a$ itself is a map of order $2q/d$ realising the positive amphichirality. It preserves $\lambda(\Pi)$, which consists of finitely many smooth arcs: the images of the edge segments of [St, Section 5.2] outside the crossing discs (smooth once the latitude function there is taken smooth, for instance affine in $h$ with small slope) and the lifted geodesic rays inside them [St, Section 5.1]. So $\lambda(\Pi)$ is smooth except at finitely many points. No non-trivial power of $\tau_a$ fixes a point of $\lambda(\Pi)$: such a power preserves the orientation of the knot, so it would fix the knot pointwise (a finite-order orientation-preserving homeomorphism of a circle with a fixed point is the identity), while the fixed set of a non-trivial isometry of $S^3$ is a (possibly empty) great sphere of dimension at most 2, on which every knot is trivial. Hence $\langle \tau_a \rangle$ permutes small disjoint balls around these points freely; smoothing the knot inside one ball of each orbit, rel a neighbourhood of the boundary (in a small ball the knot is an unknotted arc, so this is an isotopy), and transporting the change by $\langle \tau_a \rangle$ gives a smooth representative of $K$ preserved by $\tau_a$.

(ii) If an involution $f$ realises the positive amphichirality, its class has order dividing 2, and it is not the identity (it reverses the orientation of $S^3$), so $2q/d_a = 2$, i.e. $q = d_a$ is odd. Conversely, for $q$ odd take $d = q$ in (i): $\tau_a$ with $d_a = q$ is an isometric involution. No smoothness is used here.

(iii) By Lemma 2, the classes reversing both orientations are the $q$ classes $\Phi(\sigma\rho^k)$, each represented by an involution, and the positive amphichiral classes are the $q$ classes $\Phi(\tau_a)$. □

*Remark* 6. For $p$ even no element of $G$ reverses the orientation of $S^3$ [St, Remark 4.4], so $Th(p, q)$ is chiral and Problem D does not arise. The orders in (i) depend on $q$ only. The smallest one is $2^{v+1}$, where $2^v$ is the largest power of 2 dividing $q$.

## 4. The case $q = 2$

For $q = 2$ the condition $\gcd(p, q) = 1$ means that $p$ is odd, and the model of [St] is not available (its diagram has bigons). Instead we use that $K = Th(p, 2)$, $p \geq 3$ odd, is a hyperbolic amphichiral two-bridge knot with $\mathrm{Sym}(S^3, K) \cong D_4$, the dihedral group of order 8, whose orientation-preserving subgroup is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ [Q2], [MW, Theorem 4.1 and Proposition 4.2]. We write $D_4 = \langle r, s \mid r^4 = s^2 = 1,\ srs = r^{-1} \rangle$; its elements of order 4 are $r$ and $r^3$.

**Theorem 7.** *Let $p \geq 3$ be odd and $K = Th(p, 2)$.*

(i) $\mathrm{Sym}(S^3, K)$ *contains exactly two positive amphichiral classes and exactly two classes reversing both orientations. One of these two pairs is $\{r, r^3\}$, the other consists of two involutions.*

(ii) *Let $p \geq 5$. The two positive amphichiral classes are $r$ and $r^3$, of order 4, and the two classes reversing both orientations have order 2. Every finite-order homeomorphism of $S^3$ realising the positive amphichirality of a representative of $K$ has order divisible by 4. In particular $K$ is not strongly positive amphichiral, with the involution allowed to be any homeomorphism.*

(iii) (Hartley–Kawauchi) *For every odd $p \geq 3$, $K$ is strongly negative amphichiral and not strongly positive amphichiral, with involutions piecewise linear as in* [HK, p. 63].

Part (iii) is [HK, Theorem (3), p. 64]: *"Any amphicheiral 2-bridged knot is strongly negative amphicheiral but not strongly positive amphicheiral."* Let $p \geq 5$; $d = 1$ is the only odd divisor of 2, so $2q/d = 4$. By part (ii) of Theorem 7, part (ii) of Theorem 1 holds for $q = 2$, and so does the third sentence of Theorem 1 (i): every finite-order homeomorphism realising the positive amphichirality has order a multiple of 4. The first sentence of Theorem 1 (i) also holds for $q = 2$: a finite-order diffeomorphism realising the positive amphichirality has order exactly 4 (Remark 10). The second sentence is *not* claimed for $q = 2$: we do not show that a diffeomorphism or an isometry of $S^3$ of order 4 realising the positive amphichirality exists; see Remarks 9 and 10. For $p = 3$ see Remark 8.

*Proof.* (i) Write $o, \chi \colon \mathrm{Sym}(S^3, K) \to \{\pm 1\}$ for the action on the orientation of $S^3$ and of $K$; both are homomorphisms. $V = \ker o$ is the orientation-preserving subgroup $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, of index 2 since $K$ is amphichiral [Q2]. Since $K$ is (strongly) invertible [DS, §4.4, Theorem D], some class in $V$ reverses the orientation of $K$; for $p \geq 5$ this is also visible in [MW, proof of Proposition 4.4, Case 1, p. 474], where $\omega_1$ and $\omega_2$ (see (ii)) "change the orientation of the longitude". So $\chi|_V$ is onto and $N = \ker o \cap \ker \chi$ has order 2. Hence $(o, \chi)$ maps onto $\{\pm 1\}^2$, and each of its four fibres is a coset of $N$ with two elements. As a normal subgroup of order 2 of $D_4$ (it is the kernel of $(o, \chi)$), $N$ is central, so it is the centre $\{1, r^2\}$. $V$ is one of the two Klein four subgroups $\{1, r^2, s, r^2 s\}$, $\{1, r^2, rs, r^3 s\}$, and in both cases $D_4 \setminus V$ is the union

of the $N$-cosets $\{r, r^3\}$ and a pair of involutions. The positive amphichiral classes form one of these two cosets, the classes reversing both orientations the other.

(ii) Let $p \geq 5$. Then $K$ is not the figure-eight knot (its crossing number is $2(p-1) \geq 8$ [Q2]), the only knot among the words excluded from [MW, Theorem 3.6] [MW, §2], so [MW, Proposition 4.4] applies. We write $\omega_1, \omega_2, \omega_3$ for the symmetries called $\sigma_1, \sigma_2, \sigma_3$ in [MW]. On the lifted cusp triangulation of $\mathbb{R}^2$ with coordinates $(x, y)$, the deck group is $\Lambda = \mathbb{Z}(0,2) + \mathbb{Z}(4,0)$ [MW, Lemma 3.2(b)], and the meridian is vertical, the lines $x = c$, $c \in \mathbb{Z}$, being the meridional lines [MW, §3, p. 463]; so the meridian is the class of $(0,2)$, and the horizontal direction is longitudinal [MW, §3, p. 461]. By [MW, Proposition 4.4], up to deck transformations, $\omega_1$ and $\omega_2$ act as the rotations by $\pi$

$$R_1(x,y) = (2-x,\, 2-y), \qquad R_2(x,y) = (4-x,\, 2-y)$$

about $(1,1)$ and $(2,1)$, and the orientation-reversing symmetry $\omega_3$ acts as the glide reflection $\gamma(x,y) = (1-x,\, y+1)$, possibly composed with $R_1$ and $R_2$. Since $\omega_1$ and $\omega_2$ generate a subgroup $\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ of orientation-preserving symmetries [MW, proof of Proposition 4.4, p. 474], and $V$ has order 4, this subgroup is the orientation-preserving subgroup $V$; the four elements of $D_4 \backslash V = \omega_3 V$ act on the cusp torus $\mathbb{R}^2/\Lambda$ as $\gamma$, $\gamma R_1$, $\gamma R_2$, $\gamma R_1 R_2$, whichever of these $\omega_3$ itself is. Their linear parts are, in the coordinates $(x, y)$ of [MW] (first coordinate along the longitude, second along the meridian), $\mathrm{diag}(-1,1)$, $\mathrm{diag}(1,-1)$, $\mathrm{diag}(1,-1)$, $\mathrm{diag}(-1,1)$. A symmetry $f$ reversing the orientation of $S^3$ preserves the orientation of $K$ if and only if it reverses the oriented meridian $\nu$, since $\mathrm{lk}(f(\nu), f(K)) = -\mathrm{lk}(\nu, K)$; so $\gamma R_1$ and $\gamma R_2$ are the positive classes. We have $\gamma R_1(x,y) = (x-1,\, 3-y)$ and $\gamma R_2(x,y) = (x-3,\, 3-y)$, whose squares are the translations by $(-2,0)$ and $(-6,0)$. These are not in $\Lambda$, so the squares of the two positive classes act non-trivially on the cusp torus. Since $\mathrm{Sym}(S^3, K) = \mathrm{Sym}(M)$, $M = S^3 \setminus K$, acts on the lifted cusp triangulation, up to deck transformations [MW, Proposition 4.4], the squares are not the identity, and the positive classes have order 4. By (i) they are $r$ and $r^3$, and the classes reversing both orientations are the two involutions of $D_4 \setminus V$.

If a finite-order homeomorphism $f$ of order $n$ realises the positive amphichirality of a representative $K'$ of $K$, then (conjugating as in the proof of Theorem 1) its class is $r$ or $r^3$, of order 4, and this order divides $n$; so $4 \mid n$, and in particular $n \neq 2$.

(iii) This is [HK, Theorem (3), p. 64], applied to the amphichiral two-bridge knot $K$ [Q2]. □

*Remark* 8 ($p = 3$, the figure-eight knot). $Th(3,2)$ is the figure-eight knot [DS, §4.12], which is excluded from [MW, Theorem 3.6]; so the cusp argument in the proof of (ii) does not apply. If every class of order 2 in $\mathrm{Sym}(S^3, 4_1)$ that reverses the orientation of $S^3$ is represented by a piecewise linear involution of $S^3$ preserving a piecewise linear representative of the knot, then

the conclusions of (ii) hold for $p = 3$: an involution class among the positive amphichiral classes would then make $4_1$ strongly positive amphichiral in the sense of [HK], contradicting [HK, Theorem (3)], so by (i) the positive classes would be $r$ and $r^3$. We do not prove this realisation here, and we have not found the orders of the symmetries of $4_1$ in a source we have read; the paper of Kodama and Sakuma [KS] on symmetry groups of prime knots up to 10 crossings, cited in [DS, §6.3], was not available to us. Theorem 7 (i) and (iii) hold for $p = 3$ unconditionally. SnapPy finds order 4 (Remark 11).

*Remark* 9 (realisation by maps). In the proof of [MW, Proposition 4.2] the group $\mathrm{Aut}(T)$ of the canonical triangulation $T$ of $M = S^3 \setminus K$ is identified with $\mathrm{Sym}(M)$, and it is stated that it "preserves the meridian(s) of $K(\Omega)$, and therefore extends to an action on $(S^3, K(\Omega))$" [MW, pp. 471–472]. Taken literally, this gives an action of the finite group $\mathrm{Aut}(T) \cong \mathrm{Sym}(M)$ on $(S^3, K)$ by homeomorphisms, faithful because it is faithful on $M$. On this reading, for $p \geq 5$ the class $r$ is realised by a homeomorphism of order exactly 4 realising the positive amphichirality, and both classes reversing both orientations are realised by involutions, as in Theorem 1 (iii). [MW] do not describe the extension, and we draw no conclusion about smooth or piecewise linear maps from it. Independently of this reading, one class reversing both orientations is realised, for every odd $p \geq 3$, by the explicit piecewise linear involution of [HK, p. 69], "a rotation through 180° followed by a reflection in the plane of the paper". Also independently of it, $Th(p, q)$ is periodically positive amphichiral for every odd $p$ [DS, §4.4, Theorem D], which includes $q = 2$; so finite-order maps realising the positive amphichirality of $Th(p, 2)$ exist, and the divisibility statement in Theorem 7 (ii) is not vacuous. [DS] derive this from work of Kawauchi, which we have not read; the category and the order of these maps are not stated in [DS].

*Remark* 10 (diffeomorphisms). Let $p \geq 5$, let $E$ be the exterior of $K$ and $DE$ its double. Since $K$ is hyperbolic [Q2], $DE$ is a closed aspherical 3-manifold with centreless fundamental group (Lemma 4). We show: every finite-order diffeomorphism $f$ realising the positive amphichirality of a smooth representative of $K$ has order exactly 4. Indeed, let $f$ have order $n$ and its class order $m \mid n$, and suppose $m < n$. Then $h = f^m \neq \mathrm{id}$ has finite order and trivial class in $\mathrm{Sym}(S^3, K) \cong \mathrm{MCG}(E)$ [Ko, §1.3 and Proposition 1.4]. As in the proof of Lemma 5, $\langle h \rangle$ acts effectively by isometries of an averaged metric on $E$ and, by the same map on both copies, on $DE$; since $h|_E$ is isotopic to the identity, so is its double $D(h)$, which therefore induces the trivial outer automorphism of $\pi_1(DE)$. Borel's theorem [Bor], [CWY, pp. 1–2], which applies by Lemma 4 (a),(b), gives $D(h) = \mathrm{id}$, so $h$ fixes $\operatorname{int} E$ pointwise, and $h = \mathrm{id}$, being an isometry of the connected manifold $S^3$ that fixes a non-empty open set; a contradiction. So $n = m$, and $m = 4$ by Theorem 7 (ii). This remark does not provide a diffeomorphism of order 4 (compare Remark 9).

*Remark* 11 (other routes and checks). Theorem (2) of [HK] together with [Sau, Theorem 1.5] (which covers $q = 2$) also shows that $K$ is not strongly

positive amphichiral with piecewise linear involutions, for every odd $p$, and can replace [HK, Theorem (3)] in Remark 8, still with the piecewise linear realisation assumed there. Hartley and Kawauchi note that for amphichiral two-bridge knots in general the polynomial condition of their Theorem (2) does not suffice: the amphichiral two-bridge knot $(289, 251)$ of 18 crossings, in Schubert's notation, has Alexander polynomial $(4t^2 - 9t + 4)^2$ [HK, Remark (5), p. 65]. For $Th(p, 2)$ it does suffice, since $\Delta_{Th(p,2)}$ is not a Laurent square [Sau, Theorem 1.5]. The figure-eight knot $Th(3, 2) = 4_1$ [DS, §4.12] is positive amphichiral but not strongly so with piecewise linear involutions [HK, Remark (6), p. 65]. SnapPy (numerical, not certified; SnapPy 3.3.2) confirms, for odd $3 \le p \le 15$, that the two positive amphichiral isometries of the complement of $Th(p, 2)$ have order 4 and the two isometries reversing both orientations have order 2; here the orders are read from SnapPy's multiplication table of the symmetry group, and SnapPy's cusp matrices are given in the basis (meridian, longitude), so that a positive class appears there as $\mathrm{diag}(-1, 1)$, which is $\mathrm{diag}(1, -1)$ in the coordinates of [MW] used above.

## 5. Checks

The proof does not rest on computation. Two checks:

- On the combinatorial model of [St, Sections 2 and 4]: all $4q$ elements act on the directed edges of the diagram either preserving or reversing all directions; exactly $q$ elements reverse the orientation of $S^3$ and preserve that of $K$, with orders $2q/\gcd(p - 2 - 2a, q)$; exactly $q$ reverse both, of order 2; the set of orders is $\{2q/d : d \text{ odd}, d \mid q\}$, and strong positive amphichirality occurs iff $q$ is odd. 148 of 148 cases ($p$ odd, $3 \le p \le 15$, $3 \le q \le 30$, $\gcd = 1$). The signs $\mathrm{sign}(g)$ and $\varepsilon$ are taken from the table of [St, Section 4], so this check tests the action on the orientation of $K$ and the orders, not the orientation of $S^3$.
- Independently, SnapPy's isometry groups of the knot complements, with positive and negative amphichiral isometries read off from the action on the cusp (numerical, not certified): the orders of the positive amphichiral isometries are $\{2q/d : d \text{ odd}, d \mid q\}$, the negative amphichiral ones have order 2, in 63 of 63 cases ($p \in \{3, 5, 7, 9, 11\}$, $2 \le q \le 29$, $\gcd = 1$, at most 90 crossings), including the cases $q = 2$, for which see also Remark 11.

**Tool use.** This work was prepared with substantial use of Claude Code (Anthropic), an agentic AI system based on large language models, working together with the author.

- *Mathematics.* Problem D was selected by the author from a list of open problems compiled with Claude Code from the survey [DS], after the reply of O. Şavk mentioned below. The proofs were worked out by Claude Code under the author's direction.

- *Writing and typesetting.* The text and the LaTeX source were written by Claude Code, in English. The note contains no figures.
- *References.* Literature was searched with Claude Code. [HK] was read in the GDZ digitization; statements from [DS], [AFW], [Ha], [Pu], [Ko], [MW], [Lam], [Sau], [CWY], [Q2] and [St] were checked against the versions named in the bibliography; [Bor] is cited in the form stated in [CWY], and [KS] was not read (Remark 8).
- *Code.* The checks of the last section and the two checks for Section 4 were written and run by Claude Code in Python: the combinatorial model with the standard library, the isometry groups and cusp actions with SnapPy 3.3.2. The scripts and their outputs are kept with this note.
- *Checking.* Drafts were read in full by separate instances of Claude Code without access to the discussion in which the arguments were developed: twice for the text on $q \geq 3$ (5 October 2026), once for the addendum on $q = 2$ and twice for the merged text (18 October 2026), and again in full twice on 24 October 2026, the second time after the smoothing step in the proof of Theorem 1 was added; the final wording of that step was read once more on its own; another AI system, to which the author gave the text of 19 October 2026, also read it, which led to no change. Their findings were checked and incorporated where they held.

The author takes full responsibility for all contents.

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## References

[AFW] M. Aschenbrenner, S. Friedl and H. Wilton, 3-*manifold groups*, arXiv:1205.0202v3 (2013). Numbering refers to this version.

[Bor] A. Borel, *On periodic maps of certain* $K(\pi, 1)$, in: Œuvres: Collected Papers III, Springer, 1983, 57–60.

[CWY] S. Cappell, S. Weinberger and M. Yan, *Closed aspherical manifolds with center*, J. Topol. **6** (2013), 1009–1018; arXiv:1108.2321. Page numbers refer to the arXiv version v3.

[DS] A. Di Prisa and O. Şavk, *Turk's head knots and links: a survey*, Expo. Math. **44** (2026), 125777; arXiv:2409.20106. Section numbers refer to the arXiv version v3.

[Ha] A. Hatcher, *Notes on basic* 3-*manifold topology*, lecture notes, https://pi.math.cornell.edu/~hatcher/3M/3Mfds.pdf; read in the file dated 17 December 2023.

[HK] R. Hartley and A. Kawauchi, *Polynomials of amphicheiral knots*, Math. Ann. **243** (1979), 63–70; read in the digitization of the Göttinger Digitalisierungszentrum (GDZ).

[Ko] Y. Koda, *Automorphisms of the* 3-*sphere that preserve spatial graphs and handlebody-knots*, Math. Proc. Cambridge Philos. Soc. **159** (2015), 1–22; read in arXiv:1106.4777v2.

[KS] K. Kodama and M. Sakuma, *Symmetry groups of prime knots up to* 10 *crossings*, in: Knots 90 (Osaka, 1990), de Gruyter, Berlin, 1992, 323–340. Not read by us; cited after [DS].

[Lam] C. Lamm, *Strongly positive amphicheiral knots with doubly symmetric diagrams*, arXiv:2310.05106v2 (2023).

[MW] C. Millichap and W. Worden, *Hidden symmetries and commensurability of 2-bridge link complements*, Pacific J. Math. **285** (2016), 453–484.

[Pu] J. S. Purcell, *Hyperbolic knot theory*, arXiv:2002.12652v1 (2020). Numbering refers to this version.

[Q2] Werner Alois Stanggassinger, *The symmetry group of the Turk's head knots* $Th(p, 2)$, Zenodo (2026), doi:10.5281/zenodo.23562743 (version 1 of 22 October 2026, doi:10.5281/zenodo.23562744).

[Sau] S. Saurabh, *Equivariant rational sliceness and Klein amphichirality of odd-stranded Turk's head knots*, arXiv:2607.14638v1 (2026).

[St] Werner Alois Stanggassinger, *The symmetry group of Turk's head knots*, arXiv:2610.16600v1 (2026); numbering as in v1.

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