THE LEADING COEFFICIENT OF THE ALEXANDER POLYNOMIAL OF TWISTED TORUS KNOTS

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ABSTRACT. For a twisted torus knot $K(p,q;r,s)$ with $\gcd(p,q)=1$ and $2\le r<p$ we determine the leading coefficient of the Alexander polynomial Δ exactly. Writing $[p]_t\,\Delta$ up to a unit as $\widetilde{\Delta}(u)=\sum_{m=0}^{r-1}\gamma_m(t)\,u^m$ with $u=t^{rs}$, we prove (for $0<q<p$; for $q>p$ sketched in Remark 10 and checked by machine for $p\le 10$) a *multiset law* expressing each $\gamma_m\,[r]_t$ as a sum of p monomials with all coefficients $+1$, a closed formula for the degree sequence $d_m=\operatorname{top}\gamma_m$ (highest exponent) in terms of return times of a rotation, and the resulting *leading-coefficient law* $|\operatorname{lc}(\Delta)|=\#\{m : d_m+mrs \text{ maximal}\}$. Consequently Δ is monic if and only if that maximum is attained once, and Δ fails to be monic exactly when $-rs$ is an edge slope of the upper convex hull of $\{(m,d_m)\}$; in particular at most $r-1$ values of s are non-monic for fixed (p,q,r), however large $|s|$. We then classify completely, for every admissible parameter, when the r points (m,d_m) are collinear: for $0<q<p$ by three explicit conditions on the first-return map of a rotation (rendered by a proposition, proved here, as one arithmetic condition on (p,q,r) for each), for $q>p$ by the dual criterion (proved for the orbit sequence M_j; that it describes d_m for $q>p$ rests on a sketch, checked by machine for $p\le 10$), and for $q<0$ through the mirror. The extremal case $|\operatorname{lc}(\Delta)|=r$ is exactly the collinear case whose common difference st is divisible by r, at the single twist $s=-\mathrm{st}/r$ (for $q>0$). This gives an exact algebraic criterion for the monic side of a conjecture of Adnan and Park; we do not address its geometric direction.

Date: Version 8, 5 November 2026 (Proposition 21 is now proved, and in each of its parts (ii), (iii) the last condition alone is shown to be equivalent; two lemmas inserted, so that Corollary 23 of versions 4–7 is now Corollary 25; otherwise as version 7). Version 7: 4 November 2026 (the identity between Theorem 5 and Bryant's formula is now proved by an explicit bijection, Introduction; wording of that comparison sharpened; otherwise as version 6). Version 6: 4 November 2026 (citation of [Bry26] added in the Introduction and in Section 8). Version 5: 3 November 2026 (citation of [AP26c] added in the Introduction and in Section 8; in one remark "complementary range $q>p$" now reads "case $q>p$"). Version 4: 17 October 2026 (revised 19, 23, 24 and 25 October 2026). Versions 1–3: 15, 16 and 17 September 2026 (manuscript first dated 13 August 2026).

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This work was developed in collaboration with the AI system Claude Code (Anthropic): the choice of problem, the direction of the work and the computing hardware are due to the author (a sculptor); the formalisation, the algorithms and the proofs were developed by Claude Code (Anthropic) working with the author (the bijection of the Proposition in the Introduction and part of the proof of Proposition 21 are due to DeepSeek, see *Tool use*). The collaboration is stated as a fact, not a disclaimer; details are in the section on tool use at the end.

Status of the results (read first)

This is a short *computer-assisted* note. We separate carefully what is proved here, what is prior art that we cite, and what we do not touch.

- **Proved here.** The multiset law (Theorem 5), the closed degree sequence (Theorem 6), the leading-coefficient law (Theorem 7) and the convex-hull characterization (Corollary 8); further, the complete classification of the collinear case, which contains the extremal case $|\operatorname{lc}(\Delta)| = r$ (collinear with $r \mid \mathrm{st}$) — Theorem 17 for $q < p$, Theorem 19 for $q > p$ (proved for the orbit sequence M_j; that it describes d_m rests for $q > p$ on the sketch of Remark 10), Corollary 25 for $q < 0$ (for $|q| > p$ with the same proviso), and Lemma 22. Proposition 21, which renders the criteria for $0 < q < p$ as arithmetic in (p, q, r) alone, is proved in Section 6 (since version 8; part of the proof is due to DeepSeek, see *Tool use*). Section 4 gives the proof of Theorem 5 as an *outline*: its seven steps state the identities, not their derivations, and use notation that is not defined here; [St26b] writes the same objects in partly different notation. The proof is written out in full in Appendix A of the companion paper [St26b] for $0 < q < p$; Remark 10 below sketches why the same proof covers $q > p$ (a sketch, with a machine check for $p \le 10$). The identities of the outline, except the renewal identity of Step 4, have additionally been checked by machine for $3 \le p \le 7$ and all $0 < q < p$ (the README names the script and its fourteen checks); Theorem 5 itself was checked for $q > p$ as well (Section 7). Throughout, $q > 0$; negative q is reduced to positive q by mirroring (Corollary 25).
- **Prior art, cited and not claimed.** The closed formula for Δ itself is due to Park and Adnan [PA25]. The mechanism "a vertex of the Newton polytope with coefficient ± 1, uniquely maximizing in an open cone" is classical, due to Bieri and Strebel [BS81] in the form stated by Dunfield [Dun01, Thm. 4.1]. The architecture "degree bounds plus sign coherence plus corner coefficient yields a monicity characterization of a twist family" has a precedent in Hirasawa and Murasugi [HM10] for a different family.
- **Verified, not needed.** Proposition 13 evaluates the quantities of the extremal analysis without running the orbit, by Euclidean floor sums; it was checked against the direct definition on 19522 triples with no discrepancy, but it is used nowhere in the proofs of Theorems 5–7 or Corollary 8. Its status and its provenance — the machinery is classical, only the application is ours — are set out in Remark 14.
- **Not addressed.** The geometric direction of Conjecture 1.6 of [AP26], i.e. whether monicity of Δ detects fiberedness. Our criterion is exact on the Δ side only. We neither assume nor prove that conjecture.

Remark 1 (Authorship and tools). This work is joint between a sculptor and an AI coding agent; the collaboration is disclosed as a matter of fact, not as a disclaimer. All knot invariants against which our statements

are checked were computed with SageMath (`BraidGroup.burau_matrix`, `alexander_polynomial`). No knot invariant in this paper is computed by an undisclosed or bespoke algorithm. The large collinearity sweeps of Section 6 (up to 33 570 468 triples) run on our own implementations of the integer quantities of that section (orbits, return times, Birkhoff sums); they compute no knot invariant directly (the hull check of Section 7 evaluates our own law on both sides and is a check of the code). The scripts and their outputs are deposited with this version.

1. Introduction

Let $p > r \geq 2$, $q > 0$ and $\gcd(p, q) = 1$. The *twisted torus knot* $K(p, q; r, s)$ is the closure of the braid $\delta_p^q \, \delta_r^{rs}$, where $\delta_k = \sigma_1\sigma_2 \cdots \sigma_{k-1}$. For $s > 0$ these are positive braids, hence fibered with monic Alexander polynomial; the interesting regime is $s < 0$.

Park and Adnan [PA25] gave a closed formula for $\Delta_{K(p,q;r,s)}$ valid on the whole range $1 < r < p$, by Fox calculus; the earlier formula of Morton [Mo06] covers $r = 2$. They determine $\deg \Delta$ for $s > 0$ and, for $s < 0$, in two extreme regimes only ([PA25, Props. 11–13]) and write that for $s < 0$ describing the degree "becomes challenging". In [AP26] the same authors state:

Conjecture 2 ([AP26, Conj. 1.6])**.** A twisted torus knot is fibered if and only if its Alexander polynomial is monic.

Their evidence is computational (2152 knots with $s < 0$ whose standard braid diagrams have at most 100 crossings; knot Floer homology finds 49 of them non-fibered, and all 49 have non-monic Alexander polynomial), together with ten infinite families of non-monic examples ([AP26, Thms. 1.1, 1.3, 1.5]). What is missing on the algebraic side is a general law for the leading coefficient. This note supplies it.

Our statements concern $2 \leq r < p$ only. For $p > q > 0$, the range $p < r \leq p + q$, which the standard p-braid diagram does not display, has since been treated by Adnan and Park [AP26c], by Fox calculus on a genus-two Heegaard surface: they compute Δ there, prove that it is always monic, determine its breadth ([AP26c, Thms. 1.1–1.3]), and conjecture that every such knot with $r < p + q$ and $s < 0$ is fibered ([AP26c, Conj. 1.4]); by their monicity result this is the "monic implies fibered" direction of Conjecture 2, restricted to that range. In their general form that range is $\max(|p|, |q|) < r \leq |p| + |q|$; since $r < p \leq \max(p, |q|)$ in every case treated here (including $q > p$ and $q < 0$), the two parameter ranges are disjoint, and we use nothing from [AP26c].

Bryant [Bry26] has since computed Δ by a different route (Dehn surgery on the unknotted circle around the r twisted strands and a relation of Silver and Williams), with the same braid convention as ours, for all $1 < q < p$, $1 \leq r \leq p - 1$ and $s \in \mathbb{Z}$; his main result is the multivariable Alexander polynomial of the twisted torus *link*, and we use its knot case $\gcd(p, q) = 1$. There his Theorem 1.3 reads $[p]_t \, [r]_t \, \Delta \doteq \sum_{k=0}^{p-1} \sum_{i=0}^{r-1} t^{\,kq+i} \, u^{N_i(k)}$ with

$u = t^{rs}$ and $N_i(k) = \#\{1 \le m' \le k : (m'q + i) \bmod p < r\}$; he derives the span (our deg) for every s ([Bry26, Thm. 1.4]) and $|\operatorname{lc}(\Delta)|$ as the number of pairs (k, i) at which $E(k, i) = kq + i + rs\, N_i(k)$ is maximal ([Bry26, Cor. 1.5]). Grouping his sum by powers of u gives, for each m, the block $F_m(t) = \sum_{N_i(k)=m} t^{\,kq+i}$, and these blocks are our $\gamma_m \cdot [r]_t$ of Theorem 5, with no shift:

Proposition (regrouping). *Let $1 < q < p$, $\gcd(p, q) = 1$, $2 \le r < p$, $0 \le m \le r - 1$ and $j = r - 1 - m$; let $W = \{0, \dots, r - 1\}$ and w_j be as defined in the next paragraph. The map $(k, i) \mapsto v$, $v \equiv (k+1)q+i \pmod p$ with $0 \le v \le p - 1$, is a bijection from $\{(k, i) : 0 \le k \le p - 1,\ 0 \le i \le r - 1,\ N_i(k) = m\}$ onto $\{0, \dots, p-1\}$, and $kq + i = q(p-1) + v - p\, \mathrm{w}_j(v)$. Hence $F_m = \gamma_m \cdot [r]_t$ for every m; as $N_i(k) \le N_i(p-1) = r - 1$ for every pair (among $m' = 1, \dots, p-1$ the residues $m'q + i$ are all residues except i), Bryant's double sum equals $\sum_{m=0}^{r-1} u^m\, \gamma_m [r]_t$ as a polynomial.*

Proof. Fix (k, i) with $N_i(k) = m$, let $0 \le v \le p - 1$ be as above and write $(k+1)q+i = v+pL$. The sequence $x_n = v+nq$ satisfies $x_n \equiv (k+1+n)q+i \pmod p$, so for $n = 0, \dots, p-1-k$ the index $m' = k + 1 + n$ runs through $k + 1, \dots, p$. As q is invertible modulo p, the residues $m'q + i \bmod p$ for $m' = 1, \dots, p$ are all residues once, so exactly r of them lie in W; $N_i(k) = m$ of these have $m' \le k$, hence $r - m$ have $k + 1 \le m' \le p$, the last of them at $m' = p$ (residue $i \in W$). So $n = p - 1 - k$ is the $(r - m)$-th, i.e. the $(j + 1)$-st, visit of $x_n \bmod p$ to W, and $\mathrm{w}_j(v) = \lfloor x_{p-1-k}/p \rfloor$. Now $x_{p-1-k} = v + (p - 1 - k)q = (k + 1)q + i - pL + (p - 1 - k)q = p(q - L) + i$ with $0 \le i < p$, so $\mathrm{w}_j(v) = q - L$ and $q(p - 1) + v - p\, \mathrm{w}_j(v) = q(p - 1) + (k + 1)q + i - pL - p(q - L) = kq + i$. The map is injective, since v and j determine the step n of the $(j + 1)$-st visit, hence $k = p - 1 - n$ and $i = x_n \bmod p$; it is surjective, since for every v the $(j + 1)$-st visit occurs at some $n \le p - 1$ (a full period has $r \ge j + 1$ visits), and with $k = p - 1 - n$, $i = x_n \bmod p \in W$ we get $(k + 1)q + i \equiv x_n - nq \equiv v \pmod p$ and, by the count above, $N_i(k) = r - (j + 1) = m$. Therefore $F_m = \sum_{v \in \mathbb{Z}/p} t^{\,q(p-1)+v-p\, \mathrm{w}_j(v)} = \gamma_m \cdot [r]_t$ by Theorem 5. □

The bijection was found by DeepSeek in answer to our question (see *Tool use*). Version 6 of this note had, in its place, the machine comparison below, an overlap argument for large $|rs|$ giving the identity up to a unit $\pm t^e$, and a degree-shift argument giving $e = 0$; the Proposition supersedes both arguments. Two machine checks of the same combinatorial identity are kept: for every one of the 40 317 triples (p, q, r) with $p \le 60$, $1 < q < p$, $\gcd(p, q) = 1$, $2 \le r < p$, the multisets $\{kq + i : N_i(k) = m\}$ and $\{q(p - 1) + v - p\, \mathrm{w}_j(v) : 0 \le v \le p - 1\}$ (the right-hand side of Theorem 5) agree for every m (`scripts/bryant/regroup_check.py`), and the bijection, the visit step and the exponent identity are verified for the same triples (`scripts/bryant/bijection_check.py`); each script also runs a counter-case that must fail, and does. Theorem 5 itself is checked against Δ in

Sections 6 and 7. Consequently, for $1 < q < p$ and $r \geq 2$, the numerators in Theorem 5 and in [Bry26, Thm. 1.3] coincide as polynomials and the two Δ's agree up to the usual unit. The exponents within one block are pairwise distinct (if $v \neq v'$ gave the same exponent, $v - v' = p\,(\mathrm{w}_j(v) - \mathrm{w}_j(v'))$ with $0 < |v - v'| < p$), so each maximising m in Theorem 7 contributes exactly one maximising pair (k, i): Theorem 7 and [Bry26, Cor. 1.5] count the same maxima. Bryant asks when the maximum of E is attained more than once ([Bry26, Question 7.1]); for $1 < q < p$ that is the non-monic case, for which Corollary 8 gives a criterion in terms of the points (m, d_m) of Theorem 6 (a criterion, not the closed form in p, $q \bmod p$, r, s that he asks for). Our proofs do not rely on [Bry26].

Results. Up to multiplication by a unit $\pm t^n$ we may write $[p]_t\,\Delta$ as

$$\widetilde{\Delta}(u) \;=\; \sum_{m=0}^{r-1} \gamma_m(t)\, u^m, \qquad u := t^{rs}, \qquad d_m := \operatorname{top} \gamma_m, \tag{1}$$

where crucially the γ_m do *not* depend on s (Section 2). Equivalently $\widetilde{\Delta} = \det\big(I - M(P + uQ)\big)$ with the matrices of Section 2; the factor $[p]_t$ is monic and does not affect the leading coefficient. Here $\operatorname{top} f$ is the highest exponent of a Laurent polynomial f; as γ_m is in general only a quotient by $[r]_t$ (Theorem 5), we put $\operatorname{top} \gamma_m := \operatorname{top}(\gamma_m [r]_t) - (r-1)$. We write deg only for the degree of Δ in the sense of [PA25], i.e. the span (highest minus lowest exponent); the companion paper [St26b] uses the same convention. Our results are four linked statements about this decomposition. Let $W = \{0, 1, \ldots, r-1\} \subset \mathbb{Z}/p$ and, for $v \in \mathbb{Z}/p$ and $j \geq 0$, let $\mathrm{w}_j(v) := \lfloor x_n/p \rfloor$ for the integers $x_n = v + nq$ and the step $n \geq 0$ at which x_n, read modulo p, lies in W for the $(j+1)$-st time (every term in W counts, also consecutive ones), the term $x_0 = v$ itself counting if $v \in W$. Thus $\mathrm{w}_j(v)$ counts how often the orbit v, $v + q$, $v + 2q, \ldots$ wraps modulo p up to that entry (for $q > p$ a single step may wrap more than once).

(T1) a *multiset law*: each γ_m is, after multiplication by $[r]_t$, a sum of exactly p monomials, all with coefficient $+1$ (Theorem 5);

(T2) a *closed formula* for the entire degree sequence (d_m), unconditional in $\gcd(p, r)$ (Theorem 6);

(T3) the *leading-coefficient law* $|\operatorname{lc}(\Delta)| = \#\{m : d_m + mrs \text{ maximal}\}$, hence monicity iff the maximizer is unique (Theorem 7);

(C) a *convex-hull characterization* of non-monicity, with the finiteness corollary (Corollary 8).

Together, (T3) and (C) exhibit the individual non-monic families of [AP26] as instances of one statement, and give an exact monicity test costing $O(pr)$ integer operations — no polynomial arithmetic. With it, on the range $2 \leq r < p$, $s \neq 0$ treated here, Conjecture 2 becomes equivalent to the assertion "non-fibered $\iff$ $-rs$ is an edge slope of the upper hull" (for $q > p$ resting on the sketch of Remark 10).

2. Set-up and the s-free decomposition

Write $M := \beta_{\mathrm{red}}(\delta_p^q)$ and $N := \beta_{\mathrm{red}}(\delta_r^r)$ for the reduced Burau matrices, in the convention of SageMath (unreduced: $\sigma_i \mapsto I_{i-1} \oplus \left(\begin{smallmatrix} 1-t & t \\ 1 & 0 \end{smallmatrix}\right) \oplus I_{p-i-1}$; Theorem 5 is an exact identity in this convention, Theorem 7 and Corollary 8 do not depend on it), and set

$$Q := \frac{N - I}{t^r - 1} \ (\text{rank } r-1), \qquad P := I - Q.$$

Over $\mathbb{Q}(\zeta_p)(t)$ the reduced Burau matrix of δ_p is diagonalizable with eigenvalues $t\zeta^k$, $k = 1, \dots, p-1$, so M has eigenvalues $t^q\zeta^{qk}$ in the same eigenbasis; let V be an eigenbasis, each eigenvector normalised so that its first non-zero coordinate is 1, and put $G := V^{-1}QV$ (the determinants $\det G_{S,S}$ below do not depend on the normalisation; the entries of G do).

Proposition 3 (Decomposition)**.**

$$\widetilde{\Delta} \ = \sum_{\substack{S \subseteq \{1,\dots,p-1\} \\ |S| \le r-1}} (1-t^{rs})^{|S|} \prod_{j \in S} \lambda_j \prod_{j \notin S} (1-\lambda_j) \ \det\big(G_{S,S}\big), \qquad \lambda_j = t^q\zeta^{qj}.$$

Proposition 4 (Separation of parameters)**.** *G depends only on (p, r); the entire q- and s-dependence sits in the scalars λ_j and in $(1-t^{rs})$. For $r = 2$ one has $G_{jk} = \frac{1}{p}\big[(1-\zeta^j)t + (1-\zeta^{-j})\big]/(t+1)$.*

Expanding the binomials in $u = t^{rs}$ collects the s-dependence into powers of u alone and yields s-free building blocks B_m with $\widetilde{\Delta} = \sum_m (-u)^m B_m$ and $B_m = (-1)^m\gamma_m$, i.e. exactly the decomposition (1). Everything therefore reduces to the single integer sequence $(d_m)_{m=0}^{r-1}$.

3. The three theorems

Theorem 5 (Multiset law; for $q > p$ see Remark 10)**.** *For $0 \le m \le r-1$ put $j = r-1-m$. Then*

$$\gamma_m \cdot [r]_t \ = \ \sum_{v=0}^{p-1} t^{\,q(p-1)+v-p\ \mathrm{w}_j(v)}.$$

In particular the right-hand side is a sum of exactly p monomials, all with coefficient $+1$.

Theorem 6 (Degree sequence; unconditional in $\gcd(p, r)$; for $q > p$ see Remark 10)**.**

$$d_m \ = \ q(p-1) - (r-1) + \max_{0 \le v \le p-1} \big(v - p\ \mathrm{w}_{r-1-m}(v)\big).$$

The point is that the p exponents occurring in Theorem 5 are pairwise distinct modulo p, so no cancellation can occur and the top term always survives. This is what makes the degree computable by integer arithmetic alone.

Theorem 7 (Leading-coefficient law; for $q > p$ see Remark 10)**.** *For $s \neq 0$,*

$$\left|\operatorname{lc}\left(\Delta_{K(p,q;r,s)}\right)\right| \;=\; \#\big\{\, m \in \{0,\dots,r-1\} \;:\; d_m + m\,r\,s \text{ is maximal} \big\}.$$

Equivalently, Δ is monic if and only if the maximum of $m \mapsto d_m + mrs$ is attained at a single m.

Corollary 8 (Convex hull; for $q > p$ see Remark 10)**.** *$\Delta_{K(p,q;r,s)}$ fails to be monic if and only if $-rs$ is the slope of an edge of the upper convex hull of the point set $\mathcal{P} = \{(m, d_m) : 0 \le m \le r-1\}$. In that case $|\operatorname{lc}(\Delta)|$ equals the number of points of $\mathcal{P}$ lying on that edge. Consequently, for fixed (p, q, r) at most $r-1$ values of s give a non-monic Δ, no matter how large $|s|$ is.*

Remark 9. The quantity $\max_m(d_m+mrs)$ is, as a function of s, the piecewise-linear upper envelope of the r lines $s \mapsto d_m + mrs$, and non-monicity happens exactly at its integer breakpoints. This is the statement Corollary 8 supports, and it is a statement about the *top* of the Newton polygon only.

We stress what it does *not* give. The degree (equivalently the span) of Δ in the normalization of [PA25] requires the bottom of the Newton polygon as well, i.e. the valuation sequence $v_m := \operatorname{val}(\gamma_m[r]_t)$ together with the shift by $p-1 = \deg[p]_t$ coming from the factor $[p]_t$ in (1); the correct identity is $\operatorname{span}\Delta = \max_m(d_m + mrs) - \min_m(v_m + mrs) - (p-1)$, verified here on 1188 parameter tuples without discrepancy (working record; the script is not deposited). The bottom is given by Theorem 5 in the same way as the top: all coefficients there are $+1$ and the exponents are pairwise distinct modulo p, so the lowest one survives as well, and $v_m = q(p-1) + \min_{v \in \mathbb{Z}/p}\big(v - p\,\mathrm{w}_{r-1-m}(v)\big)$ (version 4 of 23 October 2026; version 3 said that no closed law for v_m was known). An earlier version of this remark claimed that Corollary 8 determines $\deg\Delta$ for all s and thereby closes the gap between [PA25, Prop. 12] and [PA25, Prop. 13]; that claim was wrong — read in the convention of [PA25] it is correct in 4 of 990 sampled cases (working record, not deposited) — and is withdrawn. The degree itself, for every $s \neq 0$ and $0 < q < p$, is determined in the companion paper [St26b]: the bottom of the Newton polygon follows from the top through the symmetry of Δ, so no law for v_m is needed there.

4. Proof of Theorem 5 (outline)

Normalizing by $\det M = \varepsilon_p\, t^{q(p-1)}$, it suffices to prove

$$\det\big(M^{-1} - P - uQ\big) \cdot [r]_t \;=\; \text{the signed orbit sum}, \tag{$\star$}$$

which we outline in seven steps, each an exact identity in $\mathbb{Q}(t,u)$. The steps state the identities; the derivations are written out in full in [St26b, Appendix A] (Step 2 in [St26b, §3, Step 3]), in partly different notation (there the bordered determinant reduces to four scalars $\xi_1,\dots,\xi_4$).

Step 1 (inverse Burau powers). With Σ the weighted down-p-cycle with wrap t^{-p},

$$A^{-n} \;=\; \Sigma^n + (t-1)t^{-p}\Big(\sum_{k<n}\Sigma^k\Big)e_{p-1}\,\omega, \qquad \Sigma^p = t^{-p}I.$$

This is an induction in which the scalar $s_n = 1 - t^{-n}$ collapses. The wrap factor is t^{-p} (not t^{1-p}); this is the step where an earlier attempt of ours went wrong, and it is checked independently.

Step 2 (full twist). For the *unreduced* $p\times p$ Burau matrix $\overline{N}$ of δ_r^r (the matrix N of Section 2 is its reduced version), $\overline{N} = t^r(I_r\oplus 0)+(1-t^r)\mathbf{1}_r\omega_r/[r]_t+(0\oplus I_{p-r})$. This is where the factor $[r]_t$ of Theorem 5 enters.

Step 3 (the pencil). Put $T(u) = X + cH_q\omega - \nu\mathbf{1}_r\omega_r$ with $X = \Sigma^q - D$. Then $\det X = (-1)^p(u^r - t^{-pq})$, and a bordering step together with a rank-two determinant lemma reduces the bordered determinant $D_B(u)$ to a few scalar quantities (in the written-out version [St26b, Appendix A, A2–A4]: four, $\xi_1,\dots,\xi_4$).

Step 4 (first-return algebra). G_C is nilpotent with $G_C^p = 0$; the renewal identity $R = Z + RG_WZ$ holds for $Z := (I - G_C)^{-1}$ (not the twist matrix N); the return matrix is $K = yK_0$ with $K_0^r = t^{-pq}I$; and the row reduction $(1-t^{-q})\,\omega Z = \omega - y^{-1}\omega_r\mathfrak{E}$ holds.

Step 5 (collapse). Three exact cancellations — first entry from outside, the γ_0 term, and the β term — yield

$$[r]_t\cdot D_B \;=\; (-1)^{p+1}\big(u^r - t^{-pq}\big)\,g(1/u), \qquad g = \frac{L + (y-1)\beta}{1-t^{-q}}, \quad \beta = \omega_r(I-K)^{-1}\mathbf{1}_W.$$

Step 6 (combinatorial identification). With $c_k := \omega_r K_0^k\mathbf{1}_W$, a renewal / first-entry argument identifies, term by term,

$$\frac{c_j - c_{j+1}}{1-t^{-q}} \;=\; \sum_{v\in\mathbb{Z}/p} t^{\,v-p\,\mathrm{w}_j(v)} \;=:\; S_j.$$

Step 7 (assembly). $\varepsilon_p(-1)^{p+1} = +1$ for both parities of p; combining Steps 1–6 gives ($\star$) and hence Theorem 5. □

Theorems 6 and 7 follow from Theorem 5: the exponents in the multiset are pairwise distinct mod p, which gives 6 without any cancellation hypothesis; and substituting $u = t^{rs}$ turns the leading coefficient into a count of maximizers of $d_m + mrs$, all of whose contributions carry the *same* sign by the coefficient-$+1$ half of Theorem 5. Corollary 8 is then plane convex geometry.

Remark 10 (The range $q > p$). The written-out proof in [St26b, Appendix A] is stated for $0 < q < p$, and that hypothesis is used at one point only: the weighted permutation matrix Σ^q has zero diagonal, so that only the identity and the full cycle contribute to $\det X$ (Step 3). For $q > p$ with $\gcd(p,q) = 1$, Σ^q is again the weighted matrix of the single p-cycle $k \mapsto k-q \bmod p$, which

has no fixed point, so its diagonal is again zero; along the full cycle the walk descends pq and the weights multiply to t^{-pq}, as before. The other steps (inverse Burau powers for every $n \geq 1$, the first-return algebra, the walk count and the assembly, where $\det M = (-t)^{q(p-1)} = (-1)^{p+1}t^{q(p-1)}$ because q is odd when p is even) and the lemmas they use hold for every braid, resp. every $q > 0$ prime to p, with the same proofs. The winding numbers $\mathrm{w}_j(v)$ are read off the integer orbit $v + nq$, so a step that wraps more than once is counted correctly. This indicates that Theorems 5–7 and Corollary 8 hold for all $q > 0$ prime to p; it is a sketch, not a written-out proof. Theorem 5 was also checked directly for $q > p$ (Section 7).

Remark 11 (Where the argument could fail, and why it does not). The passage from Theorem 5 to Theorem 7, and with it to the lattice-point count in Corollary 8, requires that the coefficients on the supporting edge be ± 1 *with coherent sign*. This is not automatic for twist families: in the tortirational family of Hirasawa and Murasugi the corner coefficient may have $|\gamma| > 1$ ([HM10, Thm. 5.5(4)]), and there the count fails. For twisted torus knots it holds, and it is exactly the content of the coefficient-$+1$ assertion of Theorem 5 — which is why that assertion is stated as part of the theorem rather than as an observation.

5. Consequences and families

An exact monicity test. Theorem 7 gives a monicity criterion whose cost is $O(pr)$ integer operations. Combined with Conjecture 2 — which we do not prove — it would be a fiberedness criterion; standing alone, it is a complete description of the leading coefficient of Δ (not of its degree; see Remark 9).

The case $q \mid r$. If $q \mid r$ then Δ is monic for every s. For $q = 1$ the closure of $\delta_p\,\delta_r^{rs}$ destabilizes to the torus knot $T(r, rs+1)$. For $q \geq 2$ these knots are cables: by Lee [Lee12, Thm. 1], for $1 < q < p$ and $1 < kq < p$ the knot $T(p, q; kq, s)$ is the $(q, p + k^2qs)$-cable over $T(k, ks+1)$, so Δ is a product of two torus-knot polynomials and monicity is visible geometrically. (Its consequence $\Delta_K(t) = \Delta_{T(k,ks+1)}(t^q)\,\Delta_{T(q,\,p+k^2qs)}(t)$ we checked against SageMath on 112 knots with $p \leq 9$, $q \geq 2$ and $s \in \{\pm 1, \pm 2\}$, without discrepancy.) This is consistent with the "fibered" side of Conjecture 2, whose geometric half we do not verify.

The collinear family. When all r points of $\mathcal{P}$ lie on one line, with common difference st, the upper hull consists of a single edge carrying every point. By Corollary 8 the maximal value $|\mathrm{lc}(\Delta)| = r$ is then attained exactly when that edge has slope $-rs$ for an integer s, i.e. when $r \mid \mathrm{st}$, at $s = -\mathrm{st}/r$; otherwise no twist reaches it. This family strictly contains the family of [AP26, Thm. 1.1]: members such as

$$T(13, 9; 3, -9), \quad T(14, 5; 3, -5), \quad T(16, 11; 3, -11), \quad T(19, 13; 3, -13)$$

have $|\operatorname{lc}| = 3$ and do not satisfy $q = p - 1$. (For the families of [AP26, Thms. 1.3, 1.5] we found $|\operatorname{lc}(\Delta)| < r$ in the cases we checked.) The collinear family is classified completely in Section 6: Theorem 17 characterizes collinearity by three explicit conditions on the first-return map of the rotation by q on the window, and gives the corresponding twist exactly.

6. The extremal case $|\operatorname{lc}(\Delta)| = r$

The extremal case $|\operatorname{lc}(\Delta)| = r$ of Corollary 8 — the upper hull is a single edge through all r points and $-rs$ is its slope — requires the points to be collinear, with common difference $\mathrm{st} = -rs$, so that $r \mid \mathrm{st}$. We classify collinearity; the extremal case is the collinear case with $r \mid \mathrm{st}$, at $s = -\mathrm{st}/r$. Collinearity deserves a closed description. It reduces as follows. Let $T\colon W \to W$ be the first-return map of the rotation by q on the window $W = \{0, \dots, r-1\} \subset \mathbb{Z}/p$, let $\rho(b)$ be its return time and $U(b)$ its winding number, so that $q\rho(b) = T(b) - b + pU(b)$.

Lemma 12. *Write $S_j(e) = \sum_{i<j} U(T^i e)$ for the Birkhoff sums of U, and put $m_j = \min_{e\in W} S_j(e)$ and $E_j = \max\{e \in W : S_j(e) = m_j\}$. Then*

$$M_j := \max_{v\in\mathbb{Z}/p} \big(v - p\ \mathrm{w}_j(v)\big) \;=\; \max_{e\in W} \big(e - p\,S_j(e)\big) \;=\; E_j - p\,m_j, \qquad \text{and} \qquad \sum_{b\in W} U(b) = q.$$

Moreover T is a single r-cycle, so $S_r \equiv q$.

Proof. For $v \notin W$ the run-up to the first entry costs $k_0 \geq 1$ steps without changing the entry points, so in the form $e_j - q\,k_j$ the value from v is strictly smaller than the value from that first entry point; hence the maximum is attained on W. For $e \in W$ the sum telescopes: $T^j(e) - q\sum_{i<j}\rho(T^i e) = T^j(e) - \sum_{i<j}\big[T^{i+1}e - T^i e + pU(T^i e)\big] = e - pS_j(e)$. Since $r < p$, every $e \leq r-1$ is smaller than p, so one extra Birkhoff unit costs more than any gain in e: the minimum of S_j decides first, the largest e among the minimizers second. Finally $\sum_b \rho(b) = p$ because the return times partition the full p-cycle, and $\sum_b (T(b) - b) = 0$ as T is a bijection, whence $qp = p\sum_b U(b)$. That the visits to W inherit the cyclic order of the p-cycle makes T an r-cycle. □

Lemma 12 still presupposes that the orbit is run: computing m_j and E_j from the definition costs $O(rp)$ steps. It need not be. Write $N_b(w)$ for the number of returns to W made by the orbit started at b within the first w wraps around $\mathbb{Z}/p$.

Proposition 13 (Euclidean evaluation)**.** *Let $r < q$. Then*

$$N_b(w) \;=\; \sum_{k=1}^{w} \left(\left\lfloor \frac{kp - b + r - 1}{q} \right\rfloor - \left\lfloor \frac{kp - b - 1}{q} \right\rfloor \right),$$

and the quantities of Lemma 12 are recovered as

$$m_j \;=\; \min\{\, w : \max_{b\in W} N_b(w) \;\geq\; j \,\}, \qquad E_j \;=\; \max\{\, b \in W :\; N_b(m_j) \;\geq\; j \,\}.$$

The sum in the first display is a difference of two floor sums $\sum_k \lfloor (ak+b)/m \rfloor$, *each evaluated in* $O(\log)$ *arithmetic operations by the Euclidean algorithm;* $N_b(\cdot)$ *is monotone, so the outer minimum admits a binary search.*

The first display is a lattice-point count: the summand is the number of integers m with $qm + b \in [kp,\, kp + r - 1]$, i.e. the number of orbit points landing in W on the k-th wrap. The second display is Lemma 12 read backwards — minimize the winding first, then take the largest admissible b.

Remark 14 (Status and provenance of Proposition 13). We record this at the level at which we have it. The identity and both recoveries were verified against the direct orbit definition on all 19522 triples with $5 \le p < 60$, $2 \le q < p$ prime to p and $2 \le r < q$, with no discrepancy, and again with the binary search in place. The proposition is not about speed: on that same small sweep the Euclidean version is in fact slower than running the orbit (69.4 s against 5.7 s), the constant being Python arithmetic per floor sum. As larger instances we obtained M_j for $(p, q, r) = (10^6 + 3,\, 999331,\, 7)$, $(10^9 + 7,\, 987654323,\, 5)$ and $(10^{12} + 39,\, 999999999989,\, 4)$ in 0.0022, 0.0043 and 0.0024 seconds respectively.

None of the machinery is ours. First-return maps of a discrete rotation to a window, Ostrowski numeration and Christoffel words are classical (for the first of these see Remark 15); the Euclidean evaluation of floor sums is standard. What is ours is only the application of that chain to the γ_m of twisted torus knots. Proposition 13 is not used in the proof of Theorems 5–7 or of Corollary 8; it makes the extremal analysis of Section 6 computable at scale, and nothing in this paper depends on it.

By Lemma 12 the points are collinear with common difference st if and only if $p\, m_j = j\, \mathrm{st} + E_j - (r-1)$ for all j; this forces $m_j = \lceil (j\, \mathrm{st} - (r-1))/p \rceil$ (necessary, not sufficient on its own): the minimal Birkhoff sum must track the line $j\, \mathrm{st}/p$ within a window of width $(r-1)/p$.

Remark 15 (Provenance of the set-up). None of the apparatus just introduced is new. The first-return map of a discrete rotation to a window of consecutive residues, together with its return-time function, is exactly the discrete form of the three-distance theorem of Sós, Surányi and Świerczkowski; Berthé and Reutenauer [BR24] treat it (in their own notation) and show that such a map is a *circular symmetric discrete interval exchange* of two or three intervals, and that the return time therefore takes at most three values. The bound $\#U(W) \le 3$ is an immediate consequence; $\Sigma_b U(b) = q$ is proved directly in Lemma 12. What we add is not the set-up but the extremal statement about M_j below.

Renormalizing the cocycle — subtracting $u_{\min} := \min U$ and setting $v := U - u_{\min}$ — makes the condition rigid; let $\widetilde{m}_j := m_j - j\, u_{\min}$ be the minimal sum of j consecutive values of v along the T-cycle, where we enumerate the cycle as $x_i := T^i(x_0)$ with indices modulo r. Writing $c := r - 1 - E_1$, the case $j = 1$ of the displayed condition gives $\mathrm{st} = p\, u_{\min} + c$, and if Z

denotes the longest run of zeros of v along the T-cycle then the condition forces $c \le (r-1)/Z$, and $c \ge p/(Z+1)$ whenever $Z+1 \le r-1$. Since $c \le r-1 < p$ this rules out $Z = 0$; and if v is supported at a single point, so that $Z = r-1$, the second bound never applies and only $c \le 1$ survives. Theorem 17 below shows that exactly three configurations remain, all of which do occur:

Proposition 16. *Let $B = \{b \in W : U(b) > u_{\min}\}$ and $k := \#B$. In each of the following cases $\{(m, d_m)\}$ is collinear, with the common difference* st *given; $|\operatorname{lc}(\Delta)| = r$ then holds at $s = -\mathrm{st}/r$ if $r \mid \mathrm{st}$, and for no s otherwise (so never in case* (K0)*, where* $\mathrm{st} = p$ *by Lemma 22 and* $\gcd(p, r) = \gcd(p, q) = 1$*):*

(K0) $B = \varnothing$*, i.e. U is constant; then* $\mathrm{st} = p\,u_{\min}$.
(K1) $B = \{b_0\}$ *with* $T(b_0) = r-1$*; then* $\mathrm{st} = p\,u_{\min}$.
(K2) $B = \{b_0\}$ *with* $b_0 = r-1$*, and T is the rotation* $e \mapsto e+1 \bmod r$*; then* $\mathrm{st} = p\,u_{\min} + 1$.

Proof. In all three cases v vanishes on a run of length $\ge r-1$ along the T-cycle, so for every $j \le r-1$ some window of length j avoids the support of v and $m_j = j\,u_{\min}$. In (K0) every e is a minimizer, so $E_j = r-1$. In (K1) the window starting one step after b_0 avoids b_0 and begins at $T(b_0) = r-1$, so again $E_j = r-1$; in both cases $M_j = (r-1) - p\,u_{\min} j$. In (K2) we may take $x_i = i$, and the window $[i, i+j)$ avoids $r-1$ exactly for $i \le r-1-j$, so $E_j = r-1-j$ and $M_j = (r-1) - j(p\,u_{\min}+1)$. Each is an arithmetic progression. □

For $(p, q, r) = (13, 9, 3)$ one finds $U = (2, 2, 5)$ and T the rotation by 1, so (K2) applies with $\mathrm{st} = 27$, recovering $T(13, 9; 3, -9)$ — a member outside the families of [AP26]. Conversely the family of [AP26, Thm. 1.1], namely $T(r|s|,\, r|s| - 1;\, r, s)$, falls under (K1) with $\mathrm{st} = r|s|$: here $p = r|s|$ and $q = p-1$, so $T(b) = b-1$ and $U(b) = 1$ for $1 \le b \le r-1$, while $T(0) = r-1$ and $U(0) = p - r > 1$; hence $B = \{0\}$ with $T(0) = r-1$, and $\mathrm{st} = p$. We also checked this by machine for all $3 \le r \le 8$ and $2 \le |s| \le 9$.

For the converse, proved in Theorem 17 below, the first step is a uniqueness observation: the interval $[jc - (r-1),\, jc]$ has length $r-1 < p$, so it contains *at most one* multiple of p; since $p\,\widetilde{m}_j$ lies in it, that multiple is determined. Hence for a collinear triple

$$(jc \bmod p) \le r-1, \qquad \widetilde{m}_j = \lfloor jc/p \rfloor, \qquad E_j = (r-1) - (jc \bmod p) \qquad (0 \le j \le r-1). \tag{2}$$

Theorem 17. *Assume $0 < q < p$ and $r \ge 3$. Then $\{(m, d_m)\}$ is collinear if and only if one of* (K0)–(K2) *holds.*

Remark 18. For $r = 2$ any two points are collinear, so the statement degenerates; it remains true (one checks that B is then always empty or a singleton, so some case always applies), but it says nothing. The hypothesis

$q < p$ is used through the elementary observation

$$\rho(b) = 1 \implies U(b) \le 1, \tag{3}$$

i.e. a single step $b \mapsto b + q$ wraps at most once, which fails otherwise: for $(p, q, r) = (3, 5, 2)$ one has $U = (3, 2)$ and $\rho = (2, 1)$. The case $q > p$ is settled by Theorem 19 below for the orbit sequence M_j; that it describes d_m there rests on the sketch of Remark 10 (the case $q = p$ is vacuous, as $\gcd(p, q) = 1$ and $p > 1$).

The proof uses one further identity. Since each return contributes $q\rho(b) - pU(b) = T(b) - b$, iterating gives $T^j(e) = e + qR_j(e) - pS_j(e)$ with $R_j(e) = \sum_{i<j} \rho(T^i e)$, so that $e - pS_j(e) = T^j(e) - qR_j(e)$. As $T^j(e) \le r - 1$ and $R_j(e) \ge j$ this yields

$$M_j \le (r - 1) - qj, \quad \text{with equality iff } T^{-i}(r - 1) \notin D \text{ for } i = 1, \dots, j, \tag{4}$$

where $D = \{b \in W : \rho(b) \ge 2\}$: equality needs some e with $T^j(e) = r - 1$, forcing $e = T^{-j}(r - 1)$, and $R_j(e) = j$, forcing $\rho \equiv 1$ along $T^{-j}(r - 1), \dots, T^{-1}(r - 1)$.

Write $w := p - r \ge 1$. Two elementary facts about the return structure will be used. First, the return times partition the p-cycle, so $\sum_{b \in W}(\rho(b) - 1) = w$; in particular $\#D \le w$, and $D = \varnothing$ can happen only for $w = 0$. Second,

$$U(b) = 0 \iff \rho(b) = 1 \text{ and } b + q \le r - 1, \tag{5}$$

because without a wrap the orbit increases strictly and can therefore never re-enter $W = [0, r)$ from above.

Proof of Theorem 17. Sufficiency is Proposition 16. For necessity, suppose $\{(m, d_m)\}$ is collinear.

Step 1: if $c \le 1$ then (K0)–(K2) *hold.* For $j \le r - 1$ we have $jc \le r - 1 < p$, so (2) gives $\widetilde{m}_j = 0$ throughout; in particular some cyclic window of length $r - 1$ has zero sum, so $k \le 1$. If $k = 0$ this is (K0). If $k = 1$, with v supported at the cycle position i_0: for $c = 0$, (2) gives $E_j = r - 1$ for all j, and at $j = r - 1$ the only surviving starting index is $i_0 + 1$, so $r - 1 = E_{r-1} = x_{i_0+1} = T(b_0)$, which is (K1). For $c = 1$, (2) gives $E_j = r - 1 - j$; at $j = 1$ the point $r - 1$ is already excluded, so $b_0 = r - 1$, and the points excluded at stage j are $x_{i_0}, \dots, x_{i_0-j+1}$, so $E_j = r - 1 - j$ forces $x_{i_0-i} = r - 1 - i$ for every i, i.e. $T(e) = e + 1$ for $e \le r - 2$ and $T(r - 1) = 0$. That is (K2).

Step 2: $c \ge 2$ forces a rigid shape. Assume $c \ge 2$. Every $e > E_1 = r - 1 - c$ satisfies $U(e) > u_{\min}$, so $k \ge c \ge 2$. Let Z be the longest run of zeros of v along the T-cycle. By (2), $\widetilde{m}_j = 0$ exactly for those $j \le r - 1$ with $jc < p$; hence $Z = \min\{r - 1, \lfloor (p - 1)/c \rfloor\}$. The value $Z = r - 1$ would mean that v vanishes on all but one point, i.e. $k \le 1$, contradicting $k \ge 2$; so $Z = \lfloor (p - 1)/c \rfloor \le r - 2$, and $Zc \le r - 1$ together with $(Z + 1)c \ge p$ gives $c \ge p - (r - 1) = w + 1$. If $u_{\min} \ge 1$, every point of the support has $U \ge 2$, hence $\rho \ge 2$ by (3), so $k \le \#D \le w < c$, contradicting $k \ge c$; therefore $u_{\min} = 0$. Now $U(E_1) = 0$ and (5) give $E_1 + q \le r - 1$, i.e. $q \le c$, while

$q = \sum_b U(b) \geq \#\{U \geq 1\} = k \geq c$; hence $q = c$. By (5) the support of U is exactly $\{b : b \geq r - q\}$, of size q, and since $\sum_b U(b) = q$ the cocycle U is $\{0,1\}$-valued. Finally, for $b \leq r - q - 1$ we get $b + q \leq r - 1$, so $\rho(b) = 1$; for $b \geq p - q$ we get $p \leq b + q$ and $b + q - p \leq r - 1$, i.e. the step wraps into W, so $\rho(b) = 1$; and for $r - q \leq b \leq p - q - 1$ we have $b + q \in [r, p - 1]$, outside W, while $b + 2q - p \in [r + q - p,\, q - 1] \subseteq W$ because $q \geq w + 1$. Hence

$$D = \{r - q,\ r - q + 1,\ \ldots,\ p - q - 1\}.$$

Step 3: the contradiction. Since $u_{\min} = 0$ we have $\mathrm{st} = p\,u_{\min} + c = q$, so collinearity means $M_j = (r - 1) - qj$ for every $j \leq r - 1$. By (4) at $j = r - 1$ this forces $T^{-i}(r - 1) \notin D$ for $i = 1, \ldots, r - 1$. But T is an r-cycle, so $\{T^{-i}(r - 1) : 1 \leq i \leq r - 1\} = W \setminus \{r - 1\}$, whence $D \subseteq \{r - 1\}$. As $w \geq 1$ we have $D \neq \varnothing$, so $D = \{r - 1\}$; comparing with Step 2, the least element of D is $r - q$, so $r - q = r - 1$ and $q = 1$ — contradicting $q = c \geq 2$. Hence $c \leq 1$ and Step 1 applies. □

The range $q > p$. Here the same scheme applies with the roles of (p, U) and (q, ρ) exchanged. Substituting $f = T^j(e)$ in $M_j = \max_e(T^j(e) - qR_j(e))$ turns it into

$$M_j = \max_{f \in W} \big(f - q\,R_j^*(f)\big), \qquad R_j^*(f) = \sum_{i=1}^{j} \rho(T^{-i} f),$$

and now it is $f \leq r - 1 < q$ that makes the dominance argument work: with $n_j = \min_f R_j^*(f)$ and $F_j = \max\{f : R_j^*(f) = n_j\}$ we get $M_j = F_j - q\,n_j$, hence (the interval $[j\,\mathrm{st} - (r-1), j\,\mathrm{st}]$ again being shorter than the modulus) collinearity is equivalent to

$$(j\,\mathrm{st} \bmod q) \leq r - 1, \qquad n_j = \lfloor j\,\mathrm{st}/q \rfloor, \qquad F_j = (r-1) - (j\,\mathrm{st} \bmod q). \quad (6)$$

Write $\rho_{\min} = \min_b \rho(b) \geq 1$ and $B^* = \{b \in W : \rho(b) > \rho_{\min}\}$.

Theorem 19. *Assume $q > p > r \geq 3$. Then the sequence (M_j) is an arithmetic progression (equivalently, by Theorem 6 and Remark 10, $\{(m, d_m)\}$ is collinear) if and only if one of the following holds:*

(K0*) $B^* = \varnothing$; *then* $\mathrm{st} = q\,\rho_{\min}$.
(K1*) $B^* = \{b_0\}$ *with* $b_0 = r - 1$; *then* $\mathrm{st} = q\,\rho_{\min}$.
(K2*) $B^* = \{b_0\}$ *with* $b_0 = 0$ *and* T *the rotation* $e \mapsto e - 1 \bmod r$; *then* $\mathrm{st} = q\,\rho_{\min} + 1$.

Proof. Taking $j = 1$ in (6) gives $n_1 = \rho_{\min}$ and hence $\mathrm{st} = q\,\rho_{\min} + c'$ with $c' := (r - 1) - F_1 \in [0, r - 1]$. Put $y := \rho - \rho_{\min} \geq 0$ and let $\widetilde{n}_j$ be the minimal sum of j consecutive values of y along the T-cycle, so that $q\,\widetilde{n}_j = j\,c' + F_j - (r - 1)$. Every $f > F_1$ satisfies $\rho(T^{-1} f) > \rho_{\min}$, so $k' := \#B^* \geq c'$.

Suppose $c' \geq 2$, and let Z' be the longest run of zeros of y. If $Z' = 0$ then $\widetilde{n}_1 \geq 1$, forcing $q \leq c' \leq r - 1 < q$; so $Z' \geq 1$. If $Z' \geq r - 1$ then y is supported at one point at most, i.e. $k' \leq 1 < 2 \leq c' \leq k'$; so $Z' \leq r - 2$ and

$j = Z' + 1$ is still in range. Now $\widetilde{n}_{Z'} = 0$ gives $Z'c' \le r - 1$ and $\widetilde{n}_{Z'+1} \ge 1$ gives $q \le (Z'+1)c' \le (r-1) + c'$, so $c' \ge q - r + 1$. Since the return times partition the p-cycle, $\sum_b y(b) = p - r\rho_{\min}$, and as each of the k' positive values is at least 1,

$$p - r \ge p - r\rho_{\min} = \sum_b y(b) \;\ge\; k' \;\ge\; c' \;\ge\; q - r + 1,$$

whence $q \le p - 1$, contradicting $q > p$. Therefore $c' \le 1$.

Then $jc' \le r - 1 < q$ for all $j \le r - 1$, so (6) gives $\widetilde{n}_j = 0$ throughout; in particular y vanishes on $r - 1$ consecutive positions and $k' \le 1$. If $k' = 0$ we are in (K0*). If $k' = 1$ with $B^* = \{b_0\}$, then $R_j^*(f)$ is minimal exactly when none of $T^{-1}f, \dots, T^{-j}f$ equals b_0, i.e. when $f \notin \{T(b_0), \dots, T^j(b_0)\}$; so $F_j = \max\big(W \smallsetminus \{T(b_0), \dots, T^j(b_0)\}\big)$. For $c' = 0$ we need $F_j = r - 1$ for all $j \le r - 1$, i.e. $r - 1 \notin \{T^i(b_0) : 1 \le i \le r - 1\} = W \smallsetminus \{b_0\}$, that is $b_0 = r - 1$, which is (K1*). For $c' = 1$ we need $F_j = r - 1 - j$, i.e. $\{T(b_0), \dots, T^j(b_0)\} = \{r - 1, \dots, r - j\}$ for every j, forcing $T^i(b_0) = r - i$, hence $T(e) = e - 1$ and $b_0 = 0$, which is (K2*). The converse follows by reading the same computation backwards. □

Remark 20. The two criteria are genuinely different, not a relabelling: applying (K0*)–(K2*) in the range $q < p$ misses 852 collinear triples with $p \le 60$. Conversely (K0)–(K2) and (K0*)–(K2*) agree on all 14 090 triples we examined with $p < q < 6p$, $p \le 25$, $r \ge 3$.

The criteria in terms of (p, q, r). Conditions (K0)–(K2) refer to T and U. They can be rewritten so that only arithmetic in (p, q, r) occurs; remarkably, for fixed (p, q) each case becomes an explicit range of r. Let $\bar{q} \in [1, p-1]$ be the inverse of q modulo p and put

$$\mu := \frac{q\bar{q} - 1}{p} \in \mathbb{Z}, \qquad \nu := q - \mu,$$

$$A^{\pm} := \min_{1 \le k < \kappa_{\pm}} (kq \bmod p), \qquad B^{\pm} := \max_{1 \le k < \kappa_{\pm}} (kq \bmod p), \qquad \kappa_+ := \bar{q}, \quad \kappa_- := p - \bar{q},$$

with the conventions $A^{\pm} = p$, $B^{\pm} = 0$ when the index range is empty. All of these depend on (p, q) only.

Proposition 21. *Let $0 < q < p$, $\gcd(p, q) = 1$ and $3 \le r < p$. Then*

(i) (K0) *holds if and only if $r = q$;*
(ii) (K2) *holds if and only if $r\mu < q$;*
(iii) (K1) *holds if and only if $r\nu < q$.*

Moreover, the form in which versions 4–7 of this note stated (ii) *and* (iii) *remains correct:* (K2) *holds if and only if $r \le A^+$, $r \le p + 1 - B^+$, $(r-1)\bar{q} \le p - 1$ and $r\mu < q$, and* (K1) *holds if and only if $r - 1 \le A^-$, $r \le p - B^-$, $(r-1)(p - \bar{q}) \le p - 1$ and $r\nu < q$; in each case the first three conditions follow from the last.*

Up to version 7 the proposition was stated in the four-condition form only, with the mechanism of its proof, and checked by machine; the proof below is new in version 8. The mechanism behind (ii) is that T is the rotation $e \mapsto e+1$ precisely when $\rho \equiv \bar{q}$ on $\{0,\dots,r-2\}$; then $U \equiv \mu$ there, and $\sum_b U(b) = q$ forces $U(r-1) = q-(r-1)\mu$, so that $r\mu < q$ is exactly $U(r-1) > \mu$, i.e. $B = \{r-1\}$. Case (iii) is the mirror statement with $e \mapsto e-1$, $\rho \equiv p-\bar{q}$ and ν in place of μ, with one difference: (K1), unlike (K2), does not require T to be a rotation; that it forces one is the "only if" part of (iii), for which we need Lemma 24. Part (i) is the following self-contained statement, which is where the constancy of the cocycle is pinned down.

Lemma 22. *U is constant on W if and only if $q = r$, and then $U \equiv 1$.*

Proof. If $q = r$, then from $b \in W$ the orbit $b, b+r, b+2r, \dots$ increases and leaves W at once; the first value that wraps, say $b+kr \geq p$, satisfies $b+kr < p+r$ because the previous value was $< p$, so $b+kr-p \in [0,r)$ lies in W. Thus the first return is the first wrap and $U \equiv 1$.

Conversely let $U \equiv u$. Summing $q\rho(b) = T(b) - b + pu$ over $b \in W$, using $\sum_b \rho(b) = p$ and $\sum_b \big(T(b)-b\big) = 0$ (as T is a bijection of W), gives $qp = rpu$, i.e. $q = ru$; in particular $u \geq 1$. The same identity gives

$$T(b) - b \;\equiv\; -pu \pmod{q} \qquad \text{for every } b \in W.$$

Since $b, T(b) \in \{0,\dots,r-1\}$, the difference $T(b)-b$ lies in $\{-(r-1),\dots,r-1\}$, a block of $2r-1$ consecutive integers. If $q \geq 2r-1$, such a block contains at most one representative of any residue class modulo q, so $T(b)-b$ would take one and the same value c_0 for all b; summing again gives $rc_0 = 0$, hence $c_0 = 0$ and $T = \mathrm{id}$, contradicting the fact that T is a single r-cycle with $r \geq 2$. Therefore $q \leq 2r-2$. If $u \geq 2$ then $q = ru \geq 2r > 2r-2$, a contradiction; so $u = 1$ and $q = r$. □

For (ii) and (iii) we need two lemmas. Throughout, $s_j := jq \bmod p \in \{1,\dots,p-1\}$ for $1 \leq j < p$.

Lemma 23. (a) *If $\mu \geq 1$, then $\lceil q/\mu \rceil \leq s_j \leq p+1-\lceil q/\mu \rceil$ for $1 \leq j < \bar{q}$.* (b) *Always $\nu \geq 1$, and $\lfloor q/\nu \rfloor \leq s_j \leq p-1-\lfloor q/\nu \rfloor$ for $1 \leq j < p-\bar{q}$.*

Proof. (a) Put $k := \bar{q}$, so $kq = 1+p\mu$ and $\gcd(k,\mu) = 1$ (a common divisor divides $kq - p\mu = 1$). Let $1 \leq j < k$ and $m := \lfloor jq/p \rfloor$. As $jq/p = \mu j/k + j/(pk)$, the fractional part of $\mu j/k$ is at most $(k-1)/k$ and $j/(pk) < 1/k$, we get $m = \lfloor \mu j/k \rfloor$, so $d := \mu j - mk$ is the residue of μj modulo k; $d \neq 0$ because $\gcd(k,\mu) = 1$ and $k \nmid j$. Hence $1 \leq d \leq k-1$, and with $p = (kq-1)/\mu$,

$$s_j = jq - pm = \frac{q\,d + m}{\mu} \;\geq\; \frac{q}{\mu}, \qquad \text{so } s_j \geq \lceil q/\mu \rceil.$$

For the upper bound, $(k-j)q \equiv 1 - s_j \pmod{p}$, and $s_j \neq 1$ (as $j \not\equiv \bar{q}$), so $s_{k-j} = p+1-s_j$; the lower bound for $k-j$ gives $s_j \leq p+1-\lceil q/\mu \rceil$.

(b) As $\bar{q} < p$ we have $\mu = (q\bar{q}-1)/p < q$, i.e. $\nu \geq 1$. Put $k' := p - \bar{q}$, so $k'q = p\nu - 1$ and $\gcd(k', \nu) = 1$ (a common divisor divides $p\nu - k'q = 1$). Let $1 \leq j < k'$ and $m := \lfloor jq/p \rfloor$; as $jq \leq (k'-1)q < p\nu$, $m \leq \nu - 1$. From $jq/p = \nu j/k' - j/(pk') < \nu j/k'$ we get $d := \nu j - mk' \geq 1$, and from $m > jq/p - 1$ we get $d < k' + j/p < k' + 1$; $d = k'$ would mean $k' \mid \nu j$, impossible, so $1 \leq d \leq k' - 1$. With $p = (qk'+1)/\nu$,

$$s_j = \frac{q\,d - m}{\nu} \;\geq\; \frac{q - \nu + 1}{\nu} \;>\; \frac{q}{\nu} - 1, \qquad \text{so } s_j \geq \lfloor q/\nu \rfloor.$$

For the upper bound, $(k'-j)q \equiv -1 - s_j \pmod{p}$ and $s_j \neq p-1$, so $s_{k'-j} = p - 1 - s_j \geq \lfloor q/\nu \rfloor$. □

The second lemma is the three-interval structure of T (Remark 15: the discrete three-distance theorem); we include a short proof in the form in which we use it, with the winding numbers. Let $\Lambda := \{(n, x) \in \mathbb{Z}^2 : nq \equiv x \pmod{p}\}$, a lattice of index p. The orbit of $b \in W$ is in W after n steps with integer displacement x if and only if $(n, x) \in \Lambda$ and $b + x \in W$; as $r - 1 < p$, for given n and b at most one such x exists. Let n_1 be the least $n \geq 1$ for which some $(n, x) \in \Lambda$ has $x \in [1, r-1]$, and δ_1 that x; let n_2 be the least $n \geq 1$ for which some $(n, x) \in \Lambda$ has $x \in [-(r-1), -1]$, and $\delta_2 := -x$. Both exist, with $n_1, n_2 \leq p-1$ ($n = \bar{q}$ gives $x = 1$, $n = p - \bar{q}$ gives $x = -1$). Define u_1, u_2 by $qn_1 = \delta_1 + pu_1$ and $qn_2 = -\delta_2 + pu_2$; then $u_1 \geq 0$ and $u_2 \geq 1$, as $pu_1 = qn_1 - \delta_1 > -p$ and $pu_2 = qn_2 + \delta_2 > 0$.

Lemma 24. (a) (n_1, δ_1) *and* $(n_2, -\delta_2)$ *form a basis of* Λ, *i.e.* $n_1\delta_2 + n_2\delta_1 = p$. (b) $\delta_1 + \delta_2 \geq r$, *so* $I_1 := [0, r - \delta_1)$, $I_2 := [\delta_2, r)$ *and* $M := [r - \delta_1, \delta_2)$ *partition* W, *with* $I_1, I_2 \neq \varnothing$, *and*

$$(T(b), \rho(b), U(b)) = \begin{cases} (b + \delta_1,\ n_1,\ u_1) & b \in I_1, \\ (b - \delta_2,\ n_2,\ u_2) & b \in I_2, \\ (b + \delta_1 - \delta_2,\ n_1 + n_2,\ u_1 + u_2) & b \in M. \end{cases}$$

Proof. Write $e_1 = (n_1, \delta_1)$, $e_2 = (n_2, -\delta_2)$. (a) As $e_1, e_2 \in \Lambda$, the lattice $\langle e_1, e_2 \rangle$ has index $(n_1\delta_2 + n_2\delta_1)/p$ in Λ, and this index is the number of points of Λ in the half-open parallelogram $\{\alpha e_1 + \beta e_2 : \alpha, \beta \in [0, 1)\}$ (a fundamental domain of $\langle e_1, e_2 \rangle$); so it suffices that this parallelogram contains no point of Λ except 0. Let (n, x) be such a point $\neq 0$; then $n > 0$ and $-\delta_2 < x < \delta_1$. If $x \geq 0$ and $n < n_1$: $x = 0$ would give $p \mid n$ with $0 < n < p$, and $x \geq 1$ contradicts the choice of n_1. If $x < 0$ and $n < n_2$, the choice of n_2 is contradicted. If $x \geq 0$ and $n \geq n_1$, then $(n, x) - e_1 = (n - n_1, x - \delta_1) \in \Lambda$ with $0 \leq n - n_1 < n_2$ and $x - \delta_1 \in [-(r-1), -1]$: for $n = n_1$ this gives $p \mid x - \delta_1$ with $0 < |x - \delta_1| < p$, otherwise it contradicts the choice of n_2. If $x < 0$ and $n \geq n_2$, then $(n, x) - e_2 = (n - n_2, x + \delta_2)$ with $0 \leq n - n_2 < n_1$ and $x + \delta_2 \in [1, r-1]$ contradicts n_1 in the same way.

(b) Let $b \in I_1$. At step n_1 the orbit lands in W at $b + \delta_1 \leq r - 1$. Suppose it lands earlier, after $1 \leq n < n_1$ steps with displacement $x \in [-b, r - 1 - b]$. Then $x \neq 0$ (as $n < p$), $x \geq 1$ contradicts the choice of n_1, and if $x < 0$

then $(n_1 - n, \delta_1 - x) \in \Lambda$ with $1 \le n_1 - n < n_1$ and $\delta_1 - x \in [1, r-1]$ (as $b \le r - 1 - \delta_1$), again a contradiction. So $\rho(b) = n_1$, $T(b) = b + \delta_1$, and $U(b) = u_1$ by $qn_1 = \delta_1 + pu_1$. The case $b \in I_2$ is the mirror image: $x = 0$ is again impossible, an earlier landing with $x \le -1$ contradicts the choice of n_2, and one with $x \in [1, r-1-b]$ gives $(n_2 - n, -\delta_2 - x) \in \Lambda$ with $-\delta_2 - x \in [-(r-1), -1]$ (as $b \ge \delta_2$). If $\delta_1 + \delta_2 < r$, a point $b \in [\delta_2, r - \delta_1) = I_1 \cap I_2$ would have $T(b) = b + \delta_1 = b - \delta_2$; hence $\delta_1 + \delta_2 \ge r$, and $|I_1| = r - \delta_1 \ge 1$, $|I_2| = r - \delta_2 \ge 1$. Finally let $b \in M$, so $r - \delta_1 \le b \le \delta_2 - 1$ and a landing needs $-\delta_2 < x < \delta_1$. Write $(n, x) = \lambda_1 e_1 + \lambda_2 e_2$ with $\lambda_1, \lambda_2 \in \mathbb{Z}$ by (a). If $\lambda_1, \lambda_2 \le 0$ then $n \le 0$; if $\lambda_1 \ge 1$ and $\lambda_2 \le 0$ then $x \ge \delta_1$; if $\lambda_1 \le 0$ and $\lambda_2 \ge 1$ then $x \le -\delta_2$. So $\lambda_1, \lambda_2 \ge 1$ and $n \ge n_1 + n_2$, with equality only for $e_1 + e_2$, which lands at $b + \delta_1 - \delta_2 \in [r - \delta_2, \delta_1 - 1] \subset W$. □

Proof of Proposition 21. Part (i) is Lemma 22. We use $\sum_b \rho(b) = p$ and $\sum_b U(b) = q$ (Lemma 12).

(ii), "if". Let $r\mu < q$. If $\mu = 0$ then $q = \bar{q} = 1$, and $b \mapsto b + 1$ with $\rho = 1$, $U = 0 = \mu$ for $b \le r - 2$. If $\mu \ge 1$, then $r \le \lfloor (q-1)/\mu \rfloor \le \lceil q/\mu \rceil - 1$, and Lemma 23(a) gives $r + 1 \le s_j \le p - r$ for $1 \le j < \bar{q}$. For $0 \le b \le r - 2$ therefore $r \le b + s_j \le p - 2$, so for $1 \le j < \bar{q}$ the orbit point after j steps is $b + s_j$, which lies outside W; at step $\bar{q}$ it lands at $b + 1 \in W$, as $b + \bar{q}q = b + 1 + p\mu$. In both cases $\rho(b) = \bar{q}$, $T(b) = b + 1$ and $U(b) = \mu$ for $b \le r - 2$; bijectivity gives $T(r-1) = 0$, and $U(r-1) = q - (r-1)\mu > \mu$, so $B = \{r-1\}$: this is (K2).

(ii), "only if". Under (K2), $T(e) = e + 1$ for $e \le r - 2$, so $q\rho(e) = 1 + pU(e)$ and $\rho(e) \equiv \bar{q} \pmod{p}$. As every return time is ≥ 1 and they sum to p, $\rho(e) \le p - r + 1 < p$, hence $\rho(e) = \bar{q}$ and $U(e) = (q\bar{q} - 1)/p = \mu$. Then $U(r-1) = q - (r-1)\mu$; as $U = \mu$ on the $r - 1 \ge 2$ points outside $B = \{r-1\}$, we have $u_{\min} = \mu$, so $B = \{r-1\}$ means $U(r-1) > \mu$, i.e. $r\mu < q$.

(iii), "if". Let $r\nu < q$, so $r \le \lfloor (q-1)/\nu \rfloor \le \lfloor q/\nu \rfloor$, and Lemma 23(b) gives $r \le s_j \le p - 1 - r$ for $1 \le j < p - \bar{q}$. For $1 \le b \le r - 1$ therefore $r + 1 \le b + s_j \le p - 2$, and at step $p - \bar{q}$ the orbit lands at $b - 1 \in W$, as $b + (p - \bar{q})q = b - 1 + p\nu$. So $\rho(b) = p - \bar{q}$, $T(b) = b - 1$ and $U(b) = \nu$ for $b \ge 1$; bijectivity gives $T(0) = r - 1$, and $U(0) = q - (r-1)\nu > \nu$, so $B = \{0\}$: this is (K1).

(iii), "only if". Assume (K1). By Lemma 24, U equals u_1 on I_1, u_2 on I_2 and $u_1 + u_2$ on M, where $u_2 \ge 1$. If $M \ne \varnothing$: for $u_1 < u_2$ the set B contains $I_2 \cup M$, for $u_1 > u_2$ it contains $I_1 \cup M$, and for $u_1 = u_2$ it equals M, so $M = \{b_0\}$ with $b_0 = r - \delta_1$, $\delta_2 = r - \delta_1 + 1$ and $T(b_0) = \delta_1 - 1 \le r - 2$; each contradicts (K1). So $M = \varnothing$, $\delta_1 + \delta_2 = r$, $u_1 \ne u_2$ (else $B = \varnothing$), and B is I_1 or I_2. $B = I_2$ would mean $\delta_1 = 1$, $b_0 = r - 1$ and $T(b_0) = 0 \ne r - 1$. Hence $B = I_1$, $\delta_2 = 1$, $b_0 = 0$, and $T(b) = b - 1$, $\rho(b) = n_2$, $U(b) = u_2$ for $1 \le b \le r - 1$. From $qn_2 = pu_2 - 1$ we get $n_2 \equiv -\bar{q} \pmod{p}$, so $n_2 = p - \bar{q}$ and $pu_2 = q(p - \bar{q}) + 1 = p(q - \mu)$, i.e. $u_2 = \nu$. Then $U(0) = q - (r-1)\nu$; as $U = \nu$ outside $B = \{0\}$, $u_{\min} = \nu$, so $U(0) > \nu$, i.e. $r\nu < q$.

The four-condition form (here $A^{\pm}$, $B^{\pm}$ are the numbers defined before the proposition, not the set B). Let $r\mu < q$. If $\mu = 0$, then $\bar{q} = 1$, $A^+ = p$, $B^+ = 0$, and all three conditions hold. If $\mu \geq 1$, the bounds $r+1 \leq s_j \leq p-r$ for $1 \leq j < \bar{q}$ give $A^+ \geq r+1$ and $p+1-B^+ \geq r+1$; and as $\rho = \bar{q}$ on the $r-1$ points $0, \dots, r-2$ and $\rho(r-1) \geq 1$, $\sum_b \rho(b) = p$ gives $(r-1)\bar{q} \leq p-1$. Likewise $r\nu < q$ gives $A^- \geq r$ and $p - B^- \geq r+1$ (trivially so if the range $1 \leq j < p-\bar{q}$ is empty) and, from $\rho = p-\bar{q}$ on $1, \dots, r-1$, $(r-1)(p-\bar{q}) \leq p-1$. Together with (ii) and (iii) this gives the four-condition equivalences. □

The proof of (ii), of the "if" part of (iii) and of Lemma 23 is due to DeepSeek, as repaired by us; DeepSeek also reduced the "only if" part of (iii) to the statement that (K1) forces $b_0 = 0$, $T(b) = b-1$ and $U = \nu$ off b_0; the proof of that statement (Lemma 24, a classical statement, Remark 15, and the case analysis) and of the four-condition form is Claude Code's (see *Tool use*). We have checked Proposition 21 in both forms, the two lemmas and the intermediate statements of the proof by machine on all 193 956 triples with $p \leq 100$ (script `check_prop_arith.py` in the folder `scripts/prop_arith/` of the deposit; the four-condition form had been checked on 63 734 triples with $p \leq 69$ for version 4), and the two halves of Lemma 22 on all 142 230 triples with $p \leq 89$ and $r \geq 2$.

Negative q. Nothing further is needed here. The knot $K(p, -q; r, -s)$, the closure of $\delta_p^{-q}\delta_r^{-rs}$, is the mirror image of $K(p, q; r, s)$ up to orientation (see [PA25, §1]; reversing every crossing gives a word whose reverse, i.e. the same closure with the opposite orientation, is conjugate to $\delta_p^{-q}\delta_r^{-rs}$); since Δ is insensitive to mirroring, the two Alexander polynomials agree up to a unit. This is the same reduction by which [PA25] adopt the convention $p > q > 0$ in the first place. Theorems 5–7 are stated for $q > 0$; for negative q we therefore do not use a sequence (d_m) of (p, q, r) at all, but transport the answer through the mirror:

Corollary 25. *Let $q < 0$. Then $|\operatorname{lc}(\Delta_{K(p,q;r,s)})| = |\operatorname{lc}(\Delta_{K(p,|q|;r,-s)})|$ for every $s \neq 0$. In particular $|\operatorname{lc}(\Delta_{K(p,q;r,s)})| = r$ if and only if $\{(m, d_m)\}$ is collinear for $(p, |q|, r)$ and $s = +\mathrm{st}/r$, where* st *is that of $(p, |q|, r)$.*

We verified the underlying identity $\Delta_{K(p,q;r,s)} = \Delta_{K(p,-q;r,-s)}$ directly in SageMath for all 1420 knots with $3 \leq p \leq 12$, $0 < q < p$, $2 \leq r < p$ and $s \in \{-3, -2, -1, 1, 2\}$, with no exception (a check of the software against a classical fact, not of the corollary).

Theorem 17 was also checked directly: for all 141 864 triples with $p \leq 90$, $q < p$ and $3 \leq r < p$ the criterion (K0)–(K2) agreed with collinearity in both directions, with the predicted value of st correct in every case (17 980 of them collinear: 2346 of type (K0), 7677 of type (K1), 7957 of type (K2)); an independent reimplementation extended this to all 33 570 468 triples with $p \leq 550$, $q < p$ and $2 \leq r < p$, again without a single disagreement, and found no collinear triple with $c \geq 2$ or $k \geq 2$. The individual steps were verified separately: (2) on the 10 059 collinear triples inside a $p \leq 70$ sweep,

the return-structure facts and (4) on 32 012 triples with $p \le 55$ and $r \ge 3$, and the two reductions of Lemma 12 on 42 026 triples ($4 \le p \le 60$, $r \ge 2$) and on 40 928 triples ($p \le 60$, $r \ge 3$).

In the sweep with $p \le 90$ above, 1873 of the 17 980 collinear triples have $r \mid \mathrm{st}$ (937 of type (K1), 936 of type (K2), none of type (K0)); these are the extremal ones.

With Lemma 22 the classification of the collinear case, and with it of the extremal case, is complete: for $0 < q < p$ Theorem 17 gives the criterion, which Proposition 21 renders as arithmetic in (p, q, r), for $q > p$ Theorem 19 gives the criterion (for the orbit sequence M_j; that it describes d_m rests for $q > p$ on the sketch of Remark 10), and for $q < 0$ Corollary 25 reduces to $|q|$; the proofs of Theorems 17 and 19, Lemma 22, Proposition 21 and Corollary 25 cover the whole parameter range. Two remarks on the shape of the answer are worth recording. First, U takes at most three distinct values, as the three-distance theorem predicts. Second, T is in general *not* a rotation of $\mathbb{Z}/r$ but a genuine exchange of three intervals — a smallest instance is $(p, q, r) = (5, 2, 4)$ with $T = [2, 3, 1, 0]$ — so the Sturmian shortcut is unavailable, which is precisely why (K2) has to name the rotation case explicitly.

The degree question of Remark 9 is settled for $0 < q < p$ in [St26b]. What remains open is the geometric side:

(1) The geometric direction of Conjecture 2. Corollary 8 (for $q > p$ resting on the sketch of Remark 10) turns it into the sharp statement: *$K(p, q; r, s)$ is non-fibered if and only if $-rs$ is an edge slope of the upper hull of $\{(m, d_m)\}$.*

7. Verification

All polynomial and matrix computations behind the note use SageMath (one auxiliary item of `cr5_checks.py`, not used by the note, uses SymPy) over exact fields ($\mathbb{Q}(t)$, $\mathbb{Q}(t, u)$, $\mathbb{Q}(\zeta_p)(t)$); the large integer sweeps of Section 6 and the hull test are plain Python and C on integers. No floating point enters any assertion. The scripts are deposited since version 4 (with additions in versions 6–8); versions 1–3 deposited only the paper. Counts below are numbers of *cases checked*, in each case with zero discrepancies.

- *Propositions 3 and 4:* 156 and 93 cases.
- *Agreement with* [PA25]*:* our closed form equals the formula of Park and Adnan in 312 cases; their formula was independently tested against SageMath in 480 cases.
- *Theorem 5:* 1228 cases with $4 \le p \le 11$ and $0 < q < p$, by an independent code path that reads γ_m off as the u-coefficient of $\det(I - M(P + uQ))$ over $K[u]$. For $q > p$ (added in version 4): all 1606 cases with $3 \le p \le 10$, $p < q \le 3p + 2$, $\gcd(p, q) = 1$, $2 \le r < p$ and every m, with zero discrepancies; as a control, the same orbit sum computed with $q \bmod p$ in place of q fails in all 1606 cases. In the same run Theorem 7 was

compared end to end with SageMath's `alexander_polynomial` for $p \le 7$, $p < q \le 2p+1$, $s \in \{-3,-2,-1,1,2\}$: 355 knots, 12 of them non-monic, no discrepancy.

- *Proof, step by step:* the script `seal_B.sage` checks fourteen identities of Section 4 as exact identities for all $3 \le p \le 7$ and all admissible (q,r): those of Steps 2–5 and 7 in 56 cases each, Step 6 in 198 cases. Of the fourteen, the sign check of Step 7 is an integer identity that cannot fail, and the polynomiality of Z in Step 4 follows from its nilpotency check; the renewal identity of Step 4 is not among them. Step 1 was checked separately in 104 cases up to $p = 10$.
- *Theorem 6:* 2412 individual degrees against freshly computed ones.
- *Theorem 7:* 1520 cases in which the number of maximizers of top $B_m + mrs$, with the degrees read off the computed B_m, equals $|\,\mathrm{lc}\,|$ as computed by SageMath; and 1064 and 468 end-to-end comparisons of $|\,\mathrm{lc}\,|$ against SageMath's `alexander_polynomial`, computed purely by integer arithmetic.
- *Corollary 8:* the hull criterion agrees with the direct count of maximizers in 27 392 320 cases with $4 \le p \le 120$ and $-80 \le s \le -1$ (of which 24 166 are non-monic); both are computed from the same d_m, so this checks the code, not the theorem. Against SageMath directly: 47 non-monic hits with 89 monic controls, and in a second, wider run 302 hits with 567 controls up to $|s| = 12$ — both directions of the equivalence, no violations.
- *Collinear family:* 41 cases plus 80 cases of the sharp direction.
- *Cable formula of* [Lee12] *(case $q \mid r$):* 112 knots, $3 \le p \le 9$, $q \ge 2$, $s \in \{\pm 1, \pm 2\}$.

The working record in which the identities of the proof outline were first checked is not deposited; the deposited script `seal_B.sage` reproduces the checks listed above.

8. Relation to the literature

- [PA25] is the source of the closed formula for Δ; our Proposition 3 agrees with it. The degree question for $s < 0$ is raised there as challenging. Corollary 8 decides monicity for every s but does not by itself determine the degree; the degree for every $s \ne 0$ and $0 < q < p$ is determined in the companion paper [St26b] (Remark 9).
- [AP26] states Conjecture 2 and gives ten infinite non-monic families, which our Corollary 8 subsumes.
- [AP26c] treats $\max(|p|,|q|) < r \le |p| + |q|$ (formula for Δ, monicity, breadth); this parameter range is disjoint from ours, and we use nothing from it.
- [Bry26] gives, for $1 < q < p$, $1 \le r \le p-1$ and all s, a closed formula for Δ (the knot case of his multivariable formula for twisted torus links), its span and its leading coefficient (as a count of pairs (k,i)), by Dehn

surgery. Grouped by powers of u, his formula is Theorem 5: the Proposition in the Introduction gives the bijection, and the agreement is also checked by machine for all 40 317 triples with $p \leq 60$. Our proofs do not rely on it.

- [Mo06] computes the Alexander polynomial for $r = 2$ and gives conditions under which a coefficient of absolute value greater than 1 occurs; we read it in full (July 2026) and use nothing from it beyond this.
- [Lee25, Lee26] concern the same family at $r = 2$. [Lee25] determines, for a negative full twist ($s = -1$), which of these knots are cable or torus knots; [Lee26, Cor. 1.2] determines their geometric types for all $s \neq 0$: apart from an explicit list of unknots, torus knots and $(2, v)$-cables, every $T(p, q; 2, s)$ with $p > q > 0$ is hyperbolic. On that list the Alexander polynomial is that of an unknot, a torus knot or a $(2, v)$-cable, hence monic, and Theorem 7 agrees: we checked 327 members of the ten families with $p \leq 31$ against the Alexander polynomial computed by Sage, without discrepancy. For the hyperbolic knots, i.e. outside that list, the classification does not provide Δ, so the case $r = 2$ of Theorem 7 does not follow from it. (An earlier version of this note, relying on the abstracts only, suggested the opposite.) Our statements are about general r with $2 \leq r < p$, and Theorem 6 is unconditional, including $\gcd(p, r) > 1$; for $r \geq 3$ we are not aware of a classification from which they would follow.
- [BS81], in the form of [Dun01, Thm. 4.1], is the classical mechanism behind Corollary 8: components of the Bieri–Neumann–Strebel invariant correspond to vertices of the Newton polytope with coefficient ± 1, a vertex contributing the cone of directions strictly maximizing it. We use the specialization picture in this spirit and claim no part of it. Compare also [McM02] for the Alexander norm and [Hir10] for specializations along lattice points of a fibered cone.
- [GL22] is the closest precedent for the counting law itself, and we claim no priority over it. For simple knots in lens spaces Greene and Luecke prove that the leading coefficient of the Alexander polynomial equals the number of times an explicit sequence attains its minimum, that the knot fibers exactly when that minimum is attained once (equivalently, when Δ is monic), and they evaluate the count by the Euclidean algorithm. Their sequence is read off directly from the multiples of q modulo p. In our setting what enters is the first-return map of that rotation to a window, and this map is in general an exchange of *three* intervals rather than a rotation — a smallest instance being $(p, q, r) = (5, 2, 4)$ — which is what makes the extremal analysis of Section 6 a separate question. To the best of our knowledge [GL22] is not cited in the literature on Alexander polynomials of twisted torus knots, and the connection seems worth drawing on its own account.
- [BR24] supplies the discrete three-distance framework used in Section 6; see Remark 15.

- [HM10] carries out the same architecture — degree bounds on coefficient polynomials, sign coherence, a corner coefficient, and then "fibered $\iff$ monic" — for torti-rational knots, under the hypothesis that the linking number ℓ of the underlying 2-bridge link is non-zero ([HM10, Thm. A]; for $\ell = 0$ they exhibit non-fibered knots with monic Alexander polynomial, [HM10, Thm. 2.4]). That is a different family, formulated without hulls or edge slopes, and there the corner may carry $|\gamma| > 1$ (see Remark 11). We regard [HM10] as the closest methodological precedent to this note.

To the best of our knowledge, the hull/edge-slope characterization of monicity for twisted torus knots, the lattice-point count for $|\operatorname{lc}(\Delta)|$, and the classification of the collinear case (and with it of the extremal case $|\operatorname{lc}(\Delta)| = r$) do not appear in the literature; in particular we found no statement anywhere that $|\operatorname{lc}(\Delta)| \le r$ for fixed r, and [AP26] describes the leading coefficients of twisted torus knots as attaining arbitrary integer values without singling out this bound. Our search covered the arXiv API (titles and abstracts), selected full texts, the citation graph (Semantic Scholar) and web search; we had no access to MathSciNet or zbMATH, so this is a statement about what we found, not a claim of non-existence. We singled out two places where the combinatorial core of Section 6 could plausibly already appear as a lemma. The survey of Alessandri and Berthé [AB98] we have now read in full: it treats three-distance phenomena, factor complexity and frequencies, but contains no statement about minima of Birkhoff sums, and in particular nothing equivalent to Theorem 17. The one item we could not obtain is Ferenczi–Holton–Zamboni's combinatorial description of the trajectories of three-interval exchanges (the second part of the series whose first part is [FHZ01]); we note that that series works throughout in the *irrational*, hence aperiodic, setting, whereas the exchange occurring here is finite and periodic — its discrete counterpart being [BR24] — so we consider an overlap unlikely, but we cannot exclude it.

Tool use

This work was prepared with substantial use of Claude Code (Anthropic), an agentic AI system based on large language models, working together with the author.

- *Mathematics.* The work on the Alexander polynomial of twisted torus knots began in July 2026 as one of three lines of work that Claude Code proposed and the author chose. The multiset law and its proof, the leading-coefficient law and the classification of the collinear case were worked out by Claude Code under the author's direction; the bijection of the Proposition in the Introduction is DeepSeek's, and so is part of the proof of Proposition 21 (below).
- *Writing and typesetting.* The text and the LaTeX source were written by Claude Code, in English. The note contains no figures.

- *References.* Literature was searched with Claude Code, including searches for prior work. Bibliographic data were checked against Crossref, arXiv abstract pages, the publishers' pages, the Zenodo records, or the sources themselves. [BS81] was not read and [McM02] not in full; [BS81] is cited in the form of [Dun01], and [McM02], like [Hir10], for comparison only.
- *Code.* The computations of Sections 6 and 7 were written and run by Claude Code, in Python, in SageMath and in C (with OpenMP). The deposited scripts are English versions of the working scripts, prepared by Claude Code; the exceptions are the check for $q > p$ (`q_gt_p/check_q_gt_p.sage`), which was written in English on 17 October 2026, and the scripts in `prop_arith/`, written in English on 5 November 2026. For every count in these two sections the README of the deposit names the script and the output line, and it states which outputs were not regenerated.
- *Checking.* Version 4 was read in full by separate instances of Claude Code, twice on 17 October 2026 and in several further rounds on 23 and 24 October 2026, without access to the discussion in which the arguments were developed; their findings were checked and incorporated where they held. The readings of 23 October corrected statements that version 3 also contained (see the version notes). Versions 5 to 8 were each read before upload by a fresh instance of Claude Code in the same way. Version 8 was in addition read from the PDF, on 5 November 2026, by two further fresh instances of Claude Code on the author's desk (one of them replaced a counter-case of `check_prop_arith.py` that could not fail, see the README), by claude.ai (two findings, both worked in: a sentence of Section 7 on the deposited scripts, and the condition $r \geq 3$ at one count) and by ChatGPT (no counterexample to Proposition 21 up to $p \leq 150$; its one objection, missing absolute-value bars in the leading-coefficient theorem, was an artefact of text extraction and changed nothing). The version-6 text of the comparison with [Bry26] was reviewed at the author's request by DeepSeek (web application, "deep think" mode, 4 November 2026), given the text in parts; asked for a direct proof of the regrouping identity, it produced the bijection that now stands in the Introduction, which we verified by hand and by machine (`scripts/bryant/bijection_check.py`). Its other findings concerned wording; those that held were incorporated in version 7, the rest (points answered by the paper itself or by the note's definitions) were not. For version 8, DeepSeek (same mode, 5 November 2026) was given a self-contained statement of Proposition 21 with our machine finding that the last condition alone suffices, and asked for a proof. It proved (ii), the "if" part of (iii) and Lemma 23, and reduced the "only if" part of (iii) to a lemma that it marked as a

gap; Claude Code proved that lemma (Lemma 24 and the case analysis). Both parts were then read by fresh instances of Claude Code, each given only the statement and the text to be read. The reading of our part found no false step; the reading of DeepSeek's found none either, but six steps stated without proof and four misleading ones, among them that the four-condition form would follow from the one-condition form without further argument. The text given here contains the repaired proof; its statements and main intermediate statements are checked by machine (`scripts/prop_arith/`). DeepSeek then refereed the text of version 8 (same mode, 5 November 2026): it found no false step and eight minor points (unproved one-line facts and terse steps), all of which are worked in.

The author takes full responsibility for all contents.

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