HOSTE'S CONDITION FOR ALL TURK'S HEAD KNOTS

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Abstract. We show that every zero z of the Alexander polynomial of every Turk's head knot $Th(p,q)$ satisfies $\operatorname{Re} z > -1$, and that every non-real zero lies on the unit circle. This answers Problem H of Di Prisa and Şavk, which asks to verify Hoste's condition for all Turk's head knots; the survey records it for $p = 3$. The proof is an exact factorisation of $\det(I - x\,S(\sigma_1\sigma_2^{-1}\sigma_3\cdots\sigma_{p-1}^{\pm 1}))$ for the reduced Burau representation S into $\lfloor (p-1)/2 \rfloor$ quadratic factors in t, whose middle coefficients involve $c_j = 4\cos^2(\pi j/p)$, and possibly one linear factor, followed by an elementary root estimate.

1. Result

For $p \geq 1$ let $A_p = \sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\cdots\sigma_{p-1}^{\pm 1} \in B_p$, i.e. $A_p = s_1 s_2 \cdots s_{p-1}$ with $s_i = \sigma_i^{(-1)^{i+1}}$ (A_1 is the trivial braid). The Turk's head link $Th(p,q)$ is the closure of A_p^q; it is alternating, and a knot if and only if $\gcd(p,q) = 1$ [DS, §2]. Hoste's condition for an alternating knot asks that every zero z of its Alexander polynomial satisfy $\operatorname{Re} z > -1$ [DS]; counterexamples among alternating knots were found by Hirasawa, Ishikawa and Suzuki [HIS19] (see also [DS, §5.3]).

Theorem 1. *Let $p, q \geq 1$ with $\gcd(p,q) = 1$. Every zero z of $\Delta_{Th(p,q)}$ satisfies $\operatorname{Re} z > -1$, and every non-real zero satisfies $|z| = 1$.*

Di Prisa and Şavk record the case $p = 3$ ("the zeroes of $\Delta_{Th(3,q)}(t)$ satisfy the Hoste's condition … Moreover, if z is a non-real root then $|z| = 1$", from AlSukaiti–Chbili [AC24]) and ask, as their Problem H, to "verify Hoste's condition for all Turk's head knots $Th(p,q)$" [DS, §5.3 and Problem H]. Theorem 1 answers it. For $p \leq 2$ or $q = 1$ the statement is classical: $Th(1,q)$ and $Th(p,1)$ are unknots, and $Th(2,q) = T(2,q)$ with q odd has $\Delta = (t^q+1)/(t+1)$, whose zeros are the q-th roots of -1 other than -1. We assume $p \geq 3$ from now on.

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This work was developed in collaboration with the AI system Claude Code (Anthropic): the choice of problem, the direction of the work and the computing hardware are due to the author (a sculptor); the formalisation, the algorithms and the proofs were developed by Claude Code (Anthropic) working with the author. The collaboration is stated as a fact, not a disclaimer; details are in the section on tool use at the end.

The engine is the following exact identity. Let $m = \lfloor (p-1)/2 \rfloor$ and $c_j = 2 + 2\cos(2\pi j/p) = 4\cos^2(\pi j/p)$ for $1 \le j \le m$; note $c_j > 0$ since $j < p/2$.

Theorem 2. *For the reduced Burau matrices S of Section 2 and every $p \ge 3$,*

$$F_p(t,x) := \det\big(I - x\,S(A_p)\big) = (1+xt)^{[p \text{ even}]}\, t^{-m} \prod_{j=1}^{m} \Big(x\,t^2 + \big(x^2 - c_j x + 1\big)t + x \Big).$$

Remark 3 (Relation to earlier work). Blackwell, Das, Mayer, Moyar, Quraishi and Stees [BDMMQS, Thm. 3.5] prove, for every spiral knot (a family containing all Turk's head knots), that Δ is a product over the nontrivial q-th roots of unity of a polynomial C_{p-1} given by a three-term recursion in the strand index. The product over roots of unity in Corollary 4 is therefore not new; what we add is a closed form in the Turk's head case (Theorem 2, obtained through the Burau representation, not by solving their recursion) and the location of the zeros. The mechanism of Section 3 — conjugating to a product of two commuting families and reducing to a relation of the form $2+\lambda+\lambda^{-1} = -\alpha^2$ with α an eigenvalue of the adjacency matrix of a graph — is that of Hironaka and Liechti [HL, Prop. 10], who prove that the Alexander polynomial of an alternating Coxeter link has only real positive zeros [HL, Thm. 3]; Theorem 2 carries the extra parameter x through this argument for the path graph, where $\alpha^2 = c_j$. At $x = -1$, i.e. $q = 2$, Theorem 2 gives only real positive zeros, in line with their theorem; we have not checked whether $Th(p,2)$ is an alternating Coxeter link in their sense. AlSukaiti and Chbili [AC24] treat $p = 3$, including the zeros. Saurabh [Sa1, Sa2] gives closed factorisations for $p = 4$ but does not discuss Hoste's condition (for $p = 4$, Hoste's condition would follow at once from the factorisation in [Sa1]); for odd p and $r \ge 1$ he proves $\Delta_{Th(p,2r)} \doteq \Delta_{Th(p,2)} H_{p,r}^2$ [Sa3, Thm. 7.4], without discussing zeros. Nasser, Mayassi and Chbili [NMC26] use the reduced Burau representation "to compute the Alexander polynomial and verify Fox's Trapezoidal Conjecture for certain families of generalized weaving links with three and four strands" and "propose an approach to addressing the conjecture for closed alternating braids by analyzing the reduced Burau representation"; we have read their abstract, not their full text.

2. Convention and Burau's theorem

$S_i := S(\sigma_i)$ is the $(p-1) \times (p-1)$ identity except row i, which has t in column $i-1$, $-t$ in column i and 1 in column $i+1$ (entries outside the matrix dropped); $S(\sigma\tau) = S(\sigma)S(\tau)$. This is the reduced Burau matrix of Birman [Bir74, (3-26)], and Burau's theorem in Birman's form [Bir74, Thm. 3.11, (3-28)] gives, for a braid $\beta \in B_p$ with knot closure,

$$\Delta_{\hat{\beta}}(t) \;\doteq\; \frac{\det\big(I - S(\beta)\big)}{1 + t + \cdots + t^{p-1}},$$

where $\doteq$ means equality up to a unit $\pm t^k$ (Birman writes $\det(\cdot - \mathrm{Id})$; this changes only the sign, and a convention differing by $t \leftrightarrow t^{-1}$ changes nothing up to a unit, since $\Delta(t^{-1}) \doteq \Delta(t)$ for a knot). Note $\det S_i = -t$, and $S_i = I + e_i r_i^{\mathsf{T}}$ with r_i supported on the columns $i-1, i, i+1$.

3. Proof of Theorem 2

Let $n = p - 1$, $O := \prod_{i \text{ odd}} \sigma_i$ and $E := \prod_{i \text{ even}} \sigma_i$ $(1 \le i \le n)$.

F1 (same parity). If $i \neq j$ have the same parity then $(r_i)_j = 0$, hence $(I + e_i r_i^{\mathsf{T}})(I + e_j r_j^{\mathsf{T}}) = I + e_i r_i^{\mathsf{T}} + e_j r_j^{\mathsf{T}}$, the factors commute, and $S(O) = I + \sum_{i \text{ odd}} e_i r_i^{\mathsf{T}}$, $S(E) = I + \sum_{i \text{ even}} e_i r_i^{\mathsf{T}}$.

F2 (conjugacy). *Let $g_1, \dots, g_n$ be elements of a group with $g_i g_j = g_j g_i$ whenever $|i - j| \ge 2$. Then all products using each g_i exactly once are conjugate.* A word orients the path $1 - 2 - \cdots - n$ (edge $\{i, i+1\}$ points away from the letter that comes first); two words with the same orientation are equal, since letters that are not adjacent commute; moving the first letter to the end, or the last to the front, is a conjugation and reverses all edges at that vertex; and every orientation of a path is reached from every other by such moves (induction on the leaf n: if vertex $n-1$ must be flipped while blocked by the edge $\{n-1, n\}$, flip the leaf n first; at the end flip n once more if the edge $\{n-1, n\}$ points the wrong way; a vertex all of whose edges point away from it can be moved to the front of the word, since the letters before it commute with it). Applied to $g_i = s_i$ in B_p: A_p is conjugate to $\big(\prod_{i \text{ odd}} s_i\big)\big(\prod_{i \text{ even}} s_i\big) = OE^{-1}$. Hence $F_p = \det(I - xS(O)S(E)^{-1}) = \det J / \det S(E)$ with $J := S(E) - xS(O)$.

F3 (tridiagonal). By F1, J is tridiagonal with

	row i odd	row i even
diagonal	$1 + xt$	$-(t + x)$
superdiagonal	$-x$	1
subdiagonal	$-xt$	t

so every product of a superdiagonal entry with the subdiagonal entry below it equals $-xt$.

F4. $\det S(E) = (-t)^{\lfloor n/2 \rfloor} = (-t)^m$.

F5 (continuant). Put $u := xt$ and let D_k be the leading principal $k \times k$ minor of J. Then $D_k = a_k D_{k-1} + u D_{k-2}$ with $D_{-1} = 0$, $D_0 = 1$, and a_k the diagonal entries, which alternate. Eliminating D_{k-1} gives $D_{k+2} = (a_{\text{odd}} a_{\text{even}} + 2u) D_k - u^2 D_{k-2}$ for $k \ge 1$. With $\mathsf{A} := xt^2 + x^2 t + t + x$ we have $a_{\text{odd}} a_{\text{even}} = -\mathsf{A}$, so $a_{\text{odd}} a_{\text{even}} + 2u = uW$ with $W := 2 - \mathsf{A}/u$.

F6 (Chebyshev). $P_k := D_{2k}/u^k$ and $Q_k := D_{2k+1}/u^k$ both satisfy $X_{k+1} = WX_k - X_{k-1}$. From $P_0 = 1$, $P_1 = (a_1 a_2 + u)/u = W - 1$, $Q_{-1} = 0$, $Q_0 = 1 + u$ we get, with U_k the Chebyshev polynomials of the second kind,

$$P_k = U_k(W/2) - U_{k-1}(W/2), \qquad Q_k = (1 + u)\, U_k(W/2).$$

F7 (zeros in W). Write $W = 2\cos\varphi$. If $p = 2m+1$, then $n = 2m$ and $U_m - U_{m-1} = \cos\big((2m+1)\varphi/2\big)/\cos(\varphi/2)$ has the m zeros $\varphi = (2k-1)\pi/p$; if $p = 2m+2$, then $n = 2m+1$ and $U_m = \sin((m+1)\varphi)/\sin\varphi$ has the zeros $\varphi = 2\pi k/p$ $(1 \le k \le m)$. Both are monic of degree m in W, hence equal $\prod_k (W - W_k)$. Since $uW = 2u - \mathsf{A}$, each factor gives $u(W - W_k) = -(\mathsf{A} - (2 - W_k)u)$, and $2 - W_k = 2 + 2\cos(\pi - \varphi_k)$ runs exactly through $\{c_j : 1 \le j \le m\}$ (put $j = (p+1)/2 - k$, resp. $j = m + 1 - k$). Therefore

$$D_n = (1+u)^{[p \text{ even}]}\,(-1)^m \prod_{j=1}^{m} \big(\mathsf{A} - c_j\, xt\big),$$

and with F4 and $\mathsf{A} - c_j xt = xt^2 + (x^2 - c_j x + 1)t + x$ this is Theorem 2. □

Corollary 4. *For* $\gcd(p,q) = 1$ *and* $\zeta = e^{2\pi i/q}$,

$$\Delta_{Th(p,q)}(t) \;\doteq\; \prod_{k=1}^{q-1} F_p\big(t, \zeta^k\big).$$

Proof. $\det(I - S(A_p)^q) = \prod_{k=0}^{q-1} \det(I - \zeta^k S(A_p)) = \prod_{k=0}^{q-1} F_p(t, \zeta^k)$. At $x = 1$ the factors of Theorem 2 are $t^2 - 2\cos(2\pi j/p)\,t + 1$, whose product with $(1+t)^{[p \text{ even}]}$ is $1 + t + \cdots + t^{p-1}$; so $F_p(t,1) = t^{-m}(1 + t + \cdots + t^{p-1})$, and Burau's theorem (Section 2) gives the claim. □

4. Proof of Theorem 1

Lemma 5. *Let* $c > 0$ *and* $\theta \in \mathbb{R}$. *Every root of* $t^2 + (2\cos\theta - c)\,t + 1$ *either has* $|t| = 1$, *or is real and positive. In both cases* $\operatorname{Re} t > -1$.

Proof. Put $a = 2\cos\theta - c < 2$. If $-2 \le a < 2$, the roots are complex conjugate (or a double root) with product 1, so $|t| = 1$ and $\operatorname{Re} t = -a/2 > -1$. If $a < -2$, the roots are real with product 1 and sum $-a > 0$, so both are positive. □

By Corollary 4 the zeros of $\Delta_{Th(p,q)}$ are the zeros in t of the factors of $F_p(t,x)$ at $x = e^{i\theta}$, $\theta = 2\pi k/q$, $1 \le k \le q-1$. For such x,

$$x\,t^2 + (x^2 - c_j x + 1)\,t + x \;=\; x\,\big(t^2 + (2\cos\theta - c_j)\,t + 1\big),$$

a quadratic with leading coefficient $x \neq 0$; since $c_j > 0$, Lemma 5 applies. For even p the factor $1 + xt$ vanishes at $t = -e^{-i\theta}$, which lies on the unit circle with $\operatorname{Re} t = -\cos\theta > -1$ because $\theta \notin 2\pi\mathbb{Z}$. The unit t^{-m} has no zeros. This proves Theorem 1. □

5. Checks

The proof does not depend on them.

- Theorem 2 and each step F1–F7 checked as exact identities for $3 \le p \le 11$; independently, both sides of Theorem 2 evaluated modulo a prime $P \equiv 1 \pmod p$ at six random points for every $3 \le p \le 200$, with 0

failures, while the variant with exponent [p odd] is rejected for every p and the variant $c_j = 2 - 2\cos(2\pi j/p)$ for every odd p (297/297; for even p the two sets of c_j coincide).
- Corollary 4 against SageMath's `Link.alexander_polynomial()` (Seifert-matrix route, independent of Burau) for all 22 Turk's head knots with $3 \le p \le 8$, $2 \le q \le 7$; the smallest real part of a zero is > -1 in all of them.
- Lemma 5 on $2 \cdot 10^5$ random (p, j, θ): no violation; the countercase $c \le 0$ fires.
- For $p = 3$, the formula of [AC24, Thm. 1.2] (stated there in the variable $s = -t$) agrees with SageMath's Alexander polynomial for the knots $q = 2, 4, 5, 7$.

The scripts and their outputs are deposited with this note.

Tool use. This work was prepared with substantial use of Claude Code (Anthropic), an agentic AI system based on large language models, working together with the author.

- *Mathematics.* Problem H was among the open problems found in a literature search carried out with Claude Code; the author chose it. The proof was worked out by Claude Code under the author's direction.
- *Writing and typesetting.* The text and the LaTeX source were written by Claude Code, in English. The note contains no figures.
- *References.* Literature was searched and sourced with Claude Code. Quotations were checked against the original texts where these were available to us (the works we have not read are named as such in the text); bibliographic data were checked against Crossref, the arXiv or the sources themselves.
- *Code.* The checks of Section 5 were written and run by Claude Code in Python, using sympy 1.14.0 and numpy 2.5.3, and SageMath 10.9 for the comparison with SageMath's Alexander polynomial. The scripts, deposited with this note at `https://doi.org/10.5281/zenodo.23154788`, are English versions of the working scripts, prepared by Claude Code; the deposited outputs agree with those of the working scripts. The deposit lists the exact software versions.
- *Checking.* Drafts were read twice by separate instances of Claude Code; their findings were checked and incorporated where they held. The proof and the cited statements were then checked once more in a fresh session against the sources.

The author takes full responsibility for all contents.

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