A CERTIFIED CENSUS OF SPIRAL KNOTS: NO SATELLITES AND NO COINCIDENCES IN A FINITE RANGE

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Abstract. The spiral knot $S(p,q,\varepsilon)$ is the closure of the braid $(\sigma_1^{\varepsilon_1}\cdots\sigma_{p-1}^{\varepsilon_{p-1}})^q$. Blackwell, Das, Mayer, Moyar, Quraishi and Stees asked whether spiral knots can be satellite knots, and when two spiral knots coincide, and they conjectured that two equal spiral knots have the same parameters (p,q) unless they are torus knots. We check all three in the range of coprime p,q with $3\le p\le 16$ and braid length $q(p-1)\le 55$, together with all coprime $3\le p\le 9$, $2\le q\le 11$: 20,428 parameter triples up to the two symmetries of the sign vector. Every one of the 20,357 triples with a non-constant sign vector gives a knot whose complement is certified hyperbolic by interval arithmetic, and any two of these knots are distinct, even up to mirror image, by a certified volume interval or by the determinant. Inside the range this settles the three questions; outside it the computation says nothing.

1. Introduction

For coprime integers $p,q\ge 2$ and a sign vector $\varepsilon=(\varepsilon_1,\dots,\varepsilon_{p-1})\in\{\pm1\}^{p-1}$, the *spiral knot* $S(p,q,\varepsilon)$ is the closure of the braid

$$(\sigma_1^{\varepsilon_1}\sigma_2^{\varepsilon_2}\cdots\sigma_{p-1}^{\varepsilon_{p-1}})^q\in B_p$$

[BDMMQS, Definition 1.1]. According to [BDMMQS, Abstract, §1 and §2], spiral knots were first defined by Brothers et al. [Bro], and Kim, Stees and Taalman studied their determinants [KST]; for $\varepsilon=(1,\dots,1)$ one gets the torus knot $T_{p,q}$, and the family also contains the weaving knots [BDMMQS, §1]. Replacing ε by $-\varepsilon$ gives the mirror image, and reversing the order of the coordinates of ε does not change the knot [BDMMQS, §2]; like [BDMMQS] we do not distinguish a knot from its mirror image. By [BDMMQS, §2

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This work was developed in collaboration with the AI system Claude Code (Anthropic): the choice of problem, the direction of the work and the computing hardware are due to the author (a sculptor); the formalisation, the algorithms and the proofs were developed by Claude Code (Anthropic) working with the author. The collaboration is stated as a fact, not a disclaimer; details are in the section on tool use at the end.

and Proposition 4.3], $S(p,q,\varepsilon)$ is a non-trivial torus knot exactly when ε is constant.

In [BDMMQS, §5] the authors write that their tables were produced "by hand", that "only knots with fairly low crossing number are included, so in particular we do not have an example of a spiral satellite knot", and ask:

Q5.1 Are all spiral knots either torus knots or hyperbolic knots? That is, do there exist spiral knots which are satellite knots?

Q5.2 When is $S(p,q,\varepsilon)$ the same knot as $S(p,q,\varepsilon')$?

C5.5 (Conjecture) If $S(p,q,\varepsilon) = S(p',q',\varepsilon')$, then either $p=p'$ and $q=q'$, or $S(p,q,\varepsilon)$ is a torus knot (so that $p=q'$ and $q=p'$).

In Q5.2 the question is asked for conditions other than reversal and negation of ε [BDMMQS, §5]. Let

$$\mathcal{R} = \big\{(p,q):\ \gcd(p,q)=1,\ p\geq 3,\ q\geq 2,\ \text{and}\\ \big(p\leq 16 \text{ and } q(p-1)\leq 55\big) \text{ or } \big(p\leq 9 \text{ and } q\leq 11\big)\big\},$$

a set of 71 pairs. We call a triple (p,q,ε) *reduced* if ε is chosen once from each class of $\{\pm 1\}^{p-1}$ under reversal and negation. (We leave out $p=2$: then ε has one entry and $S(2,q,\varepsilon)$ is a torus knot or its mirror image.)

Theorem 1. *Let (p,q,ε) and (p',q',ε') be reduced triples with $(p,q),(p',q')\in\mathcal{R}$ and ε,ε' not constant.*

(i) *$S(p,q,\varepsilon)$ is a hyperbolic knot; in particular it is not a satellite knot.*
(ii) *If $(p,q,\varepsilon)\neq(p',q',\varepsilon')$, then $S(p,q,\varepsilon)$ is neither equal to $S(p',q',\varepsilon')$ nor to its mirror image.*

Corollary 2. *For pairs in $\mathcal{R}$, the answer to Q5.1 is that there is no spiral satellite knot; for Q5.2 there is no coincidence other than reversal and negation of ε; and Conjecture C5.5 holds whenever both (p,q) and (p',q') lie in $\mathcal{R}$.*

Proof. Q5.1 is part (i) for non-constant ε; for constant ε the knot is $T_{p,q}$ or its mirror image [BDMMQS, §2]. Q5.2 and C5.5 for two non-constant sign vectors are part (ii). If ε is constant and ε' is not, the first knot is a torus knot and the second is hyperbolic, so they differ by [Pu, Theorem 8.17]. If both are constant, both knots are torus knots, and [BDMMQS, Proposition 4.3] says that $S(p,q,\varepsilon)$ and $S(q,p,\varepsilon)$ are the only spiral representations of $T_{p,q}$ (respectively of its mirror image); this is the exception allowed in C5.5. □

The 71 triples with alternating sign vector, one for each pair in $\mathcal{R}$, are the Turk's head knots $Th(p,q)$, the closures of $(\sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\cdots)^q$, also called weaving knots [DS, §2 and §3.3]; they are known to be hyperbolic [DS, §4.10, Theorem I], and to be pairwise distinct [DS, §4.1, Theorem A]. The range contains every coprime pair $p\geq 3$, $q\geq 2$ with $(p-1)(q-1)\leq 12$, the bound used for the tables of [BDMMQS, §4.3]. The largest braid length in $\mathcal{R}$ is 88 (for $(p,q)=(9,11)$).

Parts of C5.5 follow from [BDMMQS] itself: the genus of $S(p,q,\varepsilon)$ is $(p-1)(q-1)/2$ [BDMMQS, Theorem 4.2], so equal spiral knots have equal $(p-1)(q-1)$, and $S(p,q,\varepsilon) = S(q,p,\varepsilon')$ only for torus knots [BDMMQS, Theorem 4.4]. Within $\mathcal{R}$ there are 67 unordered pairs $\{(p,q),(p',q')\}$ with $(p-1)(q-1) = (p'-1)(q'-1)$ and $(p',q') \neq (q,p)$; for most of these, and for Q5.2 beyond the small cases tabulated in [BDMMQS, §4.3], the census gives new information.

2. The range

The Klein four group generated by reversal r and negation s acts on $\{\pm 1\}^n$, $n = p-1$. Its fixed points are $2^{\lceil n/2 \rceil}$ for r, none for s, and $2^{n/2}$ for rs if n is even (none if n is odd), so by Burnside's lemma there are

$$N(n) = \tfrac{1}{4}\bigl(2^n + 2^{\lceil n/2 \rceil} + [n \text{ even}]\, 2^{n/2}\bigr)$$

classes, one of which consists of the two constant vectors. Summed over $\mathcal{R}$ this gives 20,428 reduced triples, 71 of them with constant ε. Table 1 lists them by p.

p	q	pairs	reduced triples	non-constant ε
3	$2,\dots,26$	17	34	17
4	$3,\dots,17$	8	24	16
5	$2,\dots,13$	10	60	50
6	$5,\dots,11$	3	30	27
7	$2,\dots,11$	9	180	171
8	$3,\dots,11$	5	180	175
9	$2,\dots,11$	7	504	497
10	3	1	136	135
11	$2,\dots,5$	4	1,088	1,084
12	5	1	528	527
13	$2,\dots,4$	3	3,168	3,165
14	3	1	2,080	2,079
15	2	1	4,160	4,159
16	3	1	8,256	8,255
		71	20,428	20,357

Table 1. The range $\mathcal{R}$ by strand number p (q runs over the values coprime to p in the stated interval).

3. Method

For each reduced triple we build the closure of the braid word with SnapPy [SnapPy] (`Link(braid_closure=...)`), simplify the diagram, compute the determinant from the Goeritz matrix and the signature (spherogram, part of SnapPy), and pass to the knot complement. No knot table is

consulted. The 71 triples with constant ε were run as well; none of these torus knot complements was certified, as it must be.

Hyperbolicity. SnapPy's `verify_hyperbolicity` returns `True` together with complex intervals that are "certified to contain the true shapes for the hyperbolic manifold"; when it fails it returns `False`, which "does **not** certify non-hyperbolicity" [SnapPy] (its documentation compares the method with that of HIKMOT [HIKMOT]). In the installed version (module `snappy.verify.hyperbolicity`) the check first computes shape intervals that are certified, by interval methods, to contain an exact solution of the rectangular gluing equations; it then checks that these intervals have positive imaginary part and that the logarithmic edge equations and, at an unfilled cusp, the equations for the meridian and the longitude hold, so that the solution is the complete hyperbolic structure. We call it at 53 bits, randomising the triangulation between attempts, and then at 212 bits. A knot whose complement admits a complete hyperbolic structure is neither a torus knot nor a satellite knot [Pu, Theorem 8.17].

Volume. For a certified complement we compute the verified volume interval (`volume(verified=True)`) at the precision at which hyperbolicity was certified.

Lemma 3. *Let K, K' be knots with hyperbolic complements. If the certified volume intervals of the complements are disjoint, or $\det K \neq \det K'$, then K' is neither K nor the mirror image of K.*

Proof. If K' is K or its mirror image, the complements are homeomorphic, so their fundamental groups are isomorphic and, by Mostow–Prasad rigidity [Pu, Theorem 6.1], the complements are isometric and have the same volume; the true volumes lie in the certified intervals, which therefore meet. The determinant is a knot invariant and does not change under mirroring. □

Part (ii) of Theorem 1 is checked (program `resolve_pairs.py`) by sorting the certified intervals by their lower ends and examining every pair of overlapping intervals. The interval endpoints are stored as floating-point numbers rounded to nearest, so each interval is widened by a relative 10^{-12} before the comparison; the result is the same with no widening and with a widening of 10^{-9}. Numerical isometry signatures were also recorded; they are not used in the proof, and the sweep does not filter by them.

4. Results

Hyperbolicity. All 20,357 complements were certified: 20,023 at 53 bits and 334 at 212 bits. The census itself (up to 8 attempts at 53 bits and 2 at 212 bits per triple, five long-running worker processes with a memory limit of 2 GB each) left 49 triples undecided.[1] For 21 of them verification did

[1]The census was started once with a larger range, stopped after two minutes, and resumed with the range $\mathcal{R}$; the 1,590 records of the first launch lie in $\mathcal{R}$ and were kept.

not succeed within the attempts allowed (for 18 of these SnapPy's initial approximate solution had negatively oriented tetrahedra); the other 28 were killed by the memory limit of their worker (the kernel log records 28 such kills at the matching times). All 49 were run again with a fresh process for every triple, a memory limit of 4 GB, up to 40 randomisations at 53 bits and 4 at 212 bits, and, as a fallback that was never needed, cyclic rotations of the braid word; all 49 were certified on the original braid word.[2] The smallest certified volume is that of $S(3, 2, (1, -1)) = 4_1$, 2.0298832128…; the largest is that of $S(9, 11, (1, -1, 1, -1, 1, -1, 1, -1))$, about 277.6. The initial triangulations of the certified complements have between 2 and 315 tetrahedra.

Distinctness. 364 pairs of the 20,357 certified volume intervals overlap. In every one of these pairs the determinants differ. All other pairs have disjoint certified volume intervals. Lemma 3 gives part (ii) of Theorem 1. The determinant alone would not suffice: it takes 9,974 distinct values on the 20,357 knots.

Remark 4. No two certified records have the same numerical isometry signature. Different numerical signatures would not prove that two knots differ, and the proof does not use them. The verified version of the isometry signature (interval methods only) succeeded for 5,581 of the 20,357 complements; for almost all the others it stopped with an error in the verification of the canonical decomposition (`TiltInequalityNumericalVerifyError`).

5. Checks

Before the census we ran the pipeline on cases with known answers. The torus knots $T_{3,2}$, $T_{3,4}$, $T_{4,3}$ and the mirror image of $T_{5,2}$ are not certified. $S(3, 2, (1, -1)) = 4_1$ and $S(3, 5, (1, -1)) = 10_{123}$ [BDMMQS, Remark 4.5(2) and (1)] are certified. For $S(5, 3, (1, 1, -1, 1))$, the reversed and the negated vector, a cyclic rotation of the word and a Markov stabilisation give the same numerical isometry signature. The two pairs of [BDMMQS, Figures 6 and 7], which have equal Alexander polynomials and, for the pair of Figure 7, equal Jones polynomials, have equal determinants and are separated by disjoint certified volume intervals. The granny knot $3_1\#3_1$, the closure of $\sigma_1^3\sigma_2^3$, which is a satellite knot, is not certified; it was the only one of these cases reported as undecided. In the second run the same pipeline, with the changes described in Section 4, was again tested: 4_1 is certified, the granny knot is not certified for its word and any of its five non-trivial cyclic rotations, $T_{5,3}$ is not certified, and a cyclic rotation (by 16 letters, not a multiple of the

[2] An earlier attempt of this second run gave the same determinants and signatures, the same numerical volumes (relative difference below 10^{-14}) and overlapping certified volume intervals, but the program was changed while it ran (a retry after an exception in spherogram's determinant routine, which had stopped the granny counter-case of Section 5), so it is not used; the records used here come from a single run of an unchanged program (file hashes recorded at the start and unchanged afterwards), which is included with the data.

period 15) of the word of the certified triple $S(16, 3, (1, \dots, 1, -1))$ gives the same numerical isometry signature as in the census.

The enumeration was checked independently of the program that produced it: for each of the 71 pairs, the sign vectors in the output were brought to a normal form under reversal and negation and counted; each class occurs exactly once and the number of classes equals $N(p-1)$. The program that resolves the overlapping pairs examines all of them, whatever their isometry signatures; it reports a pair as unresolved when a record is duplicated, with or without a changed isometry signature, as it should. Finally, the determinant was recomputed for all 20,428 triples from the Alexander polynomial formula of [BDMMQS, Theorem 3.5] at $t = -1$, exactly, as the norm of C_{p-1} in $\mathbb{Z}[x]/(1 + x + \cdots + x^{q-1})$; it agrees with the determinant computed from the diagram in every case.

6. Scope

The theorem is about the finite range $\mathcal{R}$ only. It does not show that spiral knots are never satellites, nor prove C5.5 in general. For the 71 Turk's head knots in the range, part (i) and their distinctness as knot types are known [DS, Theorems A and I]; we include them because the computation does.

Tool use. This work was prepared with substantial use of Claude Code (Anthropic), an agentic AI system based on large language models, working together with the author.

- *Mathematics.* The questions were selected by the author from a list of open problems compiled with Claude Code. The census, the certificates and the distinctness argument were worked out by Claude Code under the author's direction.
- *Writing and typesetting.* The text and the LaTeX source were written by Claude Code, in English. The note contains one table and no figures.
- *References.* Literature was searched with Claude Code. Statements from [BDMMQS] were checked against the arXiv version v2, those from [Pu] against arXiv v1, those from [DS] against arXiv v3, and those about SnapPy against the documentation and the source code of the installed version. [Bro], [KST] and [HIKMOT] were not read; their bibliographic data were checked against Crossref or the journal's web page. The journal version of [BDMMQS] was not read.
- *Code.* The computations were written and run by Claude Code in Python 3.12.14 with SnapPy 3.3.2 and spherogram 2.4.1 (versions as probed after the runs).
- *Checking.* The draft was read by separate Claude Code agents without access to the discussion in which the arguments were developed; their findings were checked and incorporated where they held.

The author takes full responsibility for all contents.

Data

The records of all 20,428 triples (determinant, signature, certified volume interval, numerical isometry signatures), the programs and the logs are deposited in the same Zenodo record as this note (archive `package.zip`); its README names, for every number in Sections 4 and 5, the file it comes from.

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