VOL-DET, FOX, HOSTE AND A CONJECTURE OF HIRASAWA–MURASUGI FOR ALL PRIME ALTERNATING KNOTS WITH AT MOST 19 CROSSINGS

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Abstract. For every prime alternating knot with at most 19 crossings in B. A. Burton's census (51 280 976 knots, of which 51 280 967 are hyperbolic) we verify: the Vol-Det conjecture of Champanerkar, Kofman and Purcell, $\mathrm{vol}(K) < 2\pi \log\det(K)$ (for the hyperbolic ones), published for up to 16 crossings; Fox's trapezoidal conjecture in its strict form; log-concavity of the Alexander coefficients; Hoste's conjecture that every zero of the Alexander polynomial has real part > -1; and Conjecture 5.11 of Hirasawa and Murasugi on the width of the plateau. The verification is done knot by knot: Vol-Det by Adams' diagrammatic volume bound (double precision, with a margin far above rounding) and an exact determinant, with 72 remaining knots settled by verified interval volumes; the polynomial conditions on exactly computed Alexander polynomials, Hoste by an integer Routh–Hurwitz test. In particular, given the completeness of the census, a counterexample to Hoste's conjecture among prime alternating knots needs at least 20 crossings.

1. Results

Let $\mathcal{A}_{19}$ be the set of knots in the files `<c>a-hyp` and `<c>a-torus` ($3 \le c \le 19$) of the tables accompanying B. A. Burton's census of all prime knots with at most 19 crossings [Bu20]. The census is stated to be complete [Bu20]; that these files contain every alternating knot of the census is the labelling of the data, which we take over (Section 4) and which is supported by the count against OEIS A002864 below. The census lists each knot once up to mirror image; all statements below are invariant under mirroring. $\mathcal{A}_{19}$ consists of 51 280 967 hyperbolic knots and the nine torus knots $T(2, c)$, c odd. The number of knots per crossing number agrees with OEIS A002864 [OEIS] for every $c = 3, \dots, 19$ (a count, not an identification of knots). We write $\Delta_K(t) = \sum_{i=0}^{2n} a_i t^i$ up to a unit.

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This work was developed in collaboration with the AI system Claude Code (Anthropic): the choice of problem, the direction of the work and the computing hardware are due to the author (a sculptor); the formalisation, the algorithms and the proofs were developed by Claude Code (Anthropic) working with the author. The collaboration is stated as a fact, not a disclaimer; details are in the section on tool use at the end.

Theorem 1. *For every $K \in \mathcal{A}_{19}$:*

(i) (Vol-Det) *if K is hyperbolic,* $\operatorname{vol}(S^3 \setminus K) < 2\pi \log\det(K)$*;*
(ii) (Fox, strict form) $|a_0| < |a_1| < \cdots < |a_{n-r}| = \cdots = |a_{n+r}| > \cdots > |a_{2n}|$ *for some $r \geq 0$;*
(iii) (log-concavity) $|a_i|^2 \geq |a_{i-1}|\,|a_{i+1}|$ *for $0 < i < 2n$;*
(iv) (Hoste) *every zero z of Δ_K satisfies* $\operatorname{Re} z > -1$*;*
(v) (Hirasawa–Murasugi, Conj. 5.11) *if $\sigma(K) = \pm 2k$, the plateau in (ii) has half-width $r \leq k$.*

Remark 2 (What was known, and what is new). (i): Champanerkar, Kofman and Purcell stated the conjecture and verified it "for all alternating knots up to 16 crossings" [CKP16, Conj. 1.10]; S. D. Burton gives the same range [Bu18]; Egorov and Vesnin quote, with reference to [Bu20], the count of 51 280 967 alternating hyperbolic knots up to 19 crossings and still report 16 as the verified range [EV]; the K3 problem list records it as open [K3, Problem 1.29]. The conjecture is proved for several infinite families [Bu18, CKL19, EV]. New here is the complete census for $c = 17, 18, 19$ (50 789 647 knots, including those already covered by these families). (ii)–(iii): Fox's conjecture [Fox62] is reported verified for alternating knots up to 11 crossings [BK22, p. 13]; Stoimenow states that log-concavity (which implies the trapezoidal property) "is true for (knots with) alternating ... knot diagrams of at most 16 crossings" [St05], and proves both conjectures for alternating knots of genus at most 4 [St11, Thm. 9.5] (for μ-component alternating links with $\operatorname{span}\Delta + \mu \leq 9$: Thm. 9.6 there). Both are proved for further classes; see [AJK] and the survey [Chb24], whose only statement of a verification by crossing number is that "Fox checked this property for alternating knots with fewer than 12 crossings". New here is the complete census for $c = 17, 18, 19$, again including knots already covered by such classes. (iv): Hoste's conjecture [LM11, Conj. 1] is false in general [HIS19]; Stoimenow proves it up to genus 4 [St11, Thm. 9.1], and, according to [HIS19], K. Ishikawa proved it for all two-bridge knots and links (Proc. Amer. Math. Soc. 147 (2019), 2245–2254; not read by us). The conjecture is based on Hoste's own "extensive computer experiment" [LM11], whose range is not stated in the sources we read; Hirasawa and Murasugi give the largest real part of a zero among all alternating knots up to 12 crossings [HM13, Sects. 6 and 11], but do not state Hoste's inequality for these knots; beyond that we found no published verification over all alternating knots by crossing number. (v): Hirasawa and Murasugi [HM13, Conj. 5.11]; Chen proves it for alternating knots of genus two and for two-bridge knots [Ch18, Thms. 1.1, 1.2] (in that version one induction step, [Ch18, Prop. 3.8, Case 2], is written out only for odd length and the even case is called "similar"; a proof for two-bridge knots that writes out both parities is in [St26]); Azarpendar, Juhász and Kálmán prove it for diagrammatic Murasugi sums of special alternating links in which each sum has length less than three [AJK, Cor. 3.12] (with necessary conditions for equality in the case of knots, Cor. 3.16 there) and

for a class of alternating 3-braids [AJK, Thm. 3.9], and display knots with equality up to 10 crossings (their Figure 56). Among others, it is also known for alternating knots all of whose Alexander zeros are real ("stable" knots) [AJK, Sect. 3.1] and for certain closed alternating 3- and 4-braids [AJK], [Chb24, Sect. 4]. We found no published check by crossing number beyond that. "Not found" is a statement about our search (Section 4), not about unpublished work.

Remark 3 (Hoste below 20 crossings). By (iv), every prime alternating knot whose Alexander polynomial has a zero with real part ≤ -1 has at least 20 crossings (given the completeness of the census, Section 4).

2. Vol-Det

Hyperbolicity from the diagram. Lackenby states Menasco's theorem [Men84] as follows: "a link with a connected prime alternating diagram, other than the standard diagram of the $(2, n)$-torus link, is hyperbolic" [Lac03, p. 1]. For each of the 51 280 967 census diagrams we checked that it is alternating, that it is prime (no edge has the same face on both sides, no two faces share two edges) and not the standard $(2, n)$-torus diagram; all pass. Alternation is read off the Dowker–Thistlethwaite code: each crossing is passed once at an odd and once at an even position, and the sign of the even entry records whether that pass is over or under, so a diagram is alternating if and only if all entries have the same sign; in the census files `<c>a-hyp` and `<c>a-torus` every code (all 51 280 976, the nine torus knots included) consists of lowercase letters only, i.e. has no negative entry. So (i) rests neither on the census label "hyperbolic" nor on the label "alternating". The census diagrams are reduced alternating diagrams of minimal crossing number; we checked that each has $c+2$ faces with $\sum(\text{face sizes}) = 4c$ and no face of size < 2. Together with the primality test (no two faces share two edges) this excludes nugatory crossings, so the diagrams are reduced.

Upper bound. Adams [Ad15, Thm. 4.1]: for a hyperbolic alternating link with a reduced alternating projection with b_i faces of i edges, $\operatorname{vol}(S^3 \setminus L) \leq \sum_i b_i \operatorname{vol}(B_i) - a$, where B_i is the regular ideal i-bipyramid and $a = \max\{r, s\}$, with r the largest value of $\operatorname{vol}(B_p) + \operatorname{vol}(B_q)$ over pairs of non-adjacent faces (of p and q edges) and $s = \operatorname{vol}(B_p)+\operatorname{vol}(B_q)+\sum_{j=1}^{4} \big(\operatorname{vol}(B_{e_j})- V_{e_j}\big)$ over adjacent pairs (with V_e as defined there for $e \geq 3$, reading his $(n-2)v_{\text{tet}}$ as $(e-2)v_{\text{tet}}$, and $V_2 = 0$ as in his Example 4.2). We checked $\operatorname{vol}(B_e) - V_e \geq 0$ for all $2 \leq e \leq 64$; since a face of a reduced c-crossing diagram meets each crossing at most once and so has at most $c \leq 19$ edges, a is at least $\operatorname{vol}(B_p) + \operatorname{vol}(B_q)$ for the two largest faces, adjacent or not (Adams

says the same in the paragraph after his Example 4.2). We use

$$U(K) := \sum_i b_i \operatorname{vol}(B_i) - \operatorname{vol}(B_{c_1}) - \operatorname{vol}(B_{c_2}),$$

$$\operatorname{vol}(B_n) = n\Big(\Lambda\big(\tfrac{2\pi}{n}\big) + 2\Lambda\big(\tfrac{(n-2)\pi}{2n}\big)\Big),$$

c_1, c_2 the two largest face sizes and Λ the Lobachevsky function; the values $\operatorname{vol}(B_n)$, $n \le 64$, are computed with 40 digits and tabulated in double precision; $\operatorname{vol}(B_2) = 0$, $\operatorname{vol}(B_3) = 2v_{\text{tet}}$, $\operatorname{vol}(B_4) = v_{\text{oct}}$, $\operatorname{vol}(B_6) = 6.0896$ (Adams: 6.0896).

Determinant. $\det(K)$ is the absolute value of a $(c-1)$-minor of the colouring matrix read off the Dowker–Thistlethwaite code. Each row has entries $2, -1, -1$, so Hadamard gives $|\det| \le 6^{(c-1)/2} \le 1.1 \cdot 10^7$; we computed it modulo the primes $2^{31}-1$ and $2^{31}-19$ by Gaussian elimination on a GPU, lifted each residue symmetrically and required the absolute values of the two lifts to agree. This is exact.

Decision. If $U(K) + 10^{-9} < 2\pi \log\det(K)$, then (i) holds for K. This settles 51 280 895 knots; the smallest margin among them is 0.0178 (the comparison is in double precision; the smallest margin is seven orders of magnitude above the tolerance). The remaining 72 knots (4 with $c = 18$, 68 with $c = 19$) were settled by SnapPy's verified volume (interval arithmetic, 200 bits) against $2\pi \log\det$ as an interval: the decision is that the upper end of the volume interval lies below the lower end of $2\pi \log\det$, i.e. the interval for the difference $2\pi \log\det - \operatorname{vol}$ lies entirely above 0. Its smallest lower end over the 72 knots is 21.29 (at `18ah_0048491`, $\det = 1189$); the largest ratio $\operatorname{vol}/2\pi \log\det$ among them is 0.563.

c	knots	by $U(K)$	by verified volume	max $U/2\pi \log\det$
4–16	491 320	491 320	0	0.9866
17	1 769 978	1 769 978	0	0.9963
18	8 400 285	8 400 281	4	1.0203
19	40 619 384	40 619 316	68	1.0341
total	51 280 967	51 280 895	72	

The extremes over all knots (added in version 4). The smallest difference $2\pi \log\det - \operatorname{vol}$ above is a minimum over the 72 remaining knots only. To have it over all knots we computed the hyperbolic volume of every one of the 51 280 967 knots (SnapPy, double precision) and took, for each crossing number, the knot with the largest ratio V/D and the knot with the smallest difference $D - V$, where $V = \operatorname{vol}$ and $D = 2\pi \log\det$. Each of these knots was recomputed with SnapPy's verified volume (200 bits); every other knot is ranked in double precision. For 30 knots (7 with $c = 18$, 23 with $c = 19$) the first run found no positively oriented solution; a second run found one for all 30, verified, and none of them is extreme. Adams' bound $U(K)$ lies above the computed volume for every knot.

c	knots	largest V/D at	V/D	smallest $D-V$ at	$D-V$	share of 4-gons top 10	share of 4-gons all
4	1	4ah_1	0.2007	4ah_1	8.083	0.00	0.00
5	1	5ah_1	0.2313	5ah_1	9.398	0.29	0.29
6	3	6ah_3	0.3533	6ah_3	10.423	0.12	0.12
7	6	7ah_6	0.3996	7ah_4	11.358	0.19	0.19
8	18	8ah_18	0.5164	8ah_18	11.567	0.23	0.16
9	40	9ah_40	0.5536	9ah_40	12.109	0.27	0.18
10	123	10ah_122	0.5694	10ah_119	12.831	0.29	0.19
11	366	11ah_365	0.6042	11ah_363	13.223	0.35	0.20
12	1 288	12ah_1288	0.6288	12ah_1142	13.707	0.38	0.20
13	4 877	13ah_4877	0.6555	13ah_4877	14.064	0.38	0.20
14	19 536	14ah_19536	0.6720	14ah_19533	14.472	0.40	0.21
15	85 262	15ah_85262	0.6959	15ah_85259	14.437	0.42	0.21
16	379 799	16ah_379798	0.7115	16ah_379798	14.799	0.47	0.21
17	1 769 978	17ah_1769978	0.7209	17ah_1769978	15.273	0.50	0.20
18	8 400 285	18ah_8400285	0.7366	18ah_8400285	15.407	0.53	0.20
19	40 619 384	19ah_40619384	0.7444	19ah_40619384	15.818	0.53	0.20

The share of 4-gons is the fraction of quadrilateral faces in the census diagram, averaged over the ten knots with the largest V/D and over all knots of that crossing number (a random sample of 100 000 where there are more). Weaving knots $W(p,q)$, the alternating knots with the projection of the p-braid $(\sigma_1\cdots\sigma_{p-1})^q$, $c=q(p-1)$ [CKP16, p. 2], are extreme at small c: $W(3,2)=4_1$, $W(3,4)$ and $W(4,3)$ have both the largest V/D and the smallest $D-V$ at $c=4,8,9$, $W(5,4)$ at $c=16$, and $W(4,5)$ has the smallest $D-V$ at $c=15$. Elongated weaving knots are not: $W(7,3)$ ranks 90 283 by V/D among the 8 400 285 knots with 18 crossings, $W(3,7)$ ranks 3 425 among 19 536. At $c=17$ and $c=19$ there is no hyperbolic weaving knot. Every weaving knot was identified in the census by an isometry of the complements.

3. The Alexander polynomial: Fox, log-concavity, Hoste, Conjecture 5.11

Exact Δ_K. From the Wirtinger presentation (over- and underarcs and crossing signs, read with Regina) we form the Alexander matrix M_0+tM_1, take a $(c-1)$-minor, evaluate it at $t=2,\dots,c+1$ modulo $2^{31}-1$ on the GPU, and interpolate. The minor has degree $\le c-1$ in t; on $|t|=1$ each row has Euclidean norm at most $\sqrt{6}$, so every coefficient is bounded by $6^{(c-1)/2}<(2^{31}-1)/2$, and the symmetric lift is exact. For every knot we checked $\Delta_K(1)=\pm1$, the symmetry $|a_i|=|a_{2n-i}|$, alternating signs, and $|\Delta_K(-1)|=\det(K)$ against the independent colouring computation of Section 2.

Fox and log-concavity. (ii) and (iii) are integer comparisons on the $|a_i|$.

Hoste. $\operatorname{Re} z > -1$ for all zeros z of Δ means that all zeros of $Q(w) := \Delta(w-1)$ lie in the open right half-plane, i.e. that $R(w) := Q(-w)$ is a Hurwitz polynomial. We decide this by the Routh–Hurwitz test in fraction-free integer arithmetic, and, after multiplying R by -1 if its leading coefficient is negative (Δ is defined only up to $\pm t^k$), report "holds" only if all first-column entries are strictly positive; so a zero on the line $\operatorname{Re} z = -1$ can never be reported as satisfying the conjecture. There were no violations and no degenerate (boundary) cases.

Conjecture 5.11. In Hirasawa–Murasugi's formulation [HM13, Conj. 5.11], for alternating K with $\sigma(K) = 2k$ the coefficients increase strictly up to a plateau of half-width $m \le k$ (for $\sigma = 0$, no plateau). By (ii) the strict increase holds; a plateau with $m \ge 1$ occurs for 24 277 knots of $\mathcal{A}_{19}$ (24 268 hyperbolic ones and the nine torus knots), and for each of them we computed $\sigma(K)$ with SnapPy/spherogram and found $|\sigma(K)| \ge 2m$, with equality in 2 120 cases (2 111 hyperbolic ones and the nine torus knots).

Torus knots. The nine $T(2,c)$ run through the same chain: $\Delta = (t^c + 1)/(t+1)$ satisfies (ii)–(iv), and (v) with equality.

4. Calibration and limits

Calibration before the runs. GPU determinants and Alexander polynomials against spherogram/Sage on 1 269 census knots ($c = 4, \dots, 19$): 1269/1269; two deliberate errors are caught: a wrong colouring rule gives a different determinant for 116 of the 120 knots with 12 crossings among the 1 269, on which it was tested, and ignoring the crossing signs gives the right Alexander polynomial for only 91 of the 1 269 knots. $U(K) \ge$ SnapPy volume on the same 1 269 knots. The Hoste test on 7 hand-made polynomials (among them a zero exactly at -1, reported as a boundary case, and zeros at -0.9 and -1.1), on 2 634 random polynomials against numerical roots, and on the relevant factor (degree 114) of the Alexander polynomial of the Hirasawa–Ishikawa–Suzuki counterexample $L(51, 14, 64, 1)$ [HIS19], which it rejects. The coefficient tests on 9 hand-made cases, each violation detected exactly where planted. Our run confirms Vol-Det for all 491 320 hyperbolic knots with $c \le 16$, the range published in [CKP16].

What is taken over. Completeness of Burton's census (every prime alternating knot with ≤ 19 crossings appears), supported by the count against OEIS A002864. Minimality of the census diagrams (used only for crossing numbers; Adams' bound needs reducedness, which we checked). The statement of Menasco's theorem is taken from Lackenby's paper; Menasco's original we have not read.

Prior-art search. Web search and the sources cited, 25 September – 4 October 2026; for Conjecture 5.11 in addition Semantic Scholar citation lists and zbMATH Open. Read only as the author's preprint, not in the journal version: Stoimenow, *Hoste's conjecture and roots of link polynomials* (Ann. Comb., 2018). Not read beyond its abstract: Nasser–Mayassi–Chbili,

Attacks on Fox's trapezoidal conjecture (Vietnam J. Math., 2026). Fox's paper [Fox62] we have not read; the conjecture is quoted from [Ch18, BK22]. Statements cited from arXiv versions were not compared with the journal versions.

Deposit. Deposited with this note are the code, the summaries and logs of all runs, and per-knot files for the exceptions: the 72 knots settled by verified volume, the 24 277 plateau knots, and for each c the knot with the smallest margin. The per-knot arrays over all 51 280 967 hyperbolic knots (about 8 GB) are not included; they are reproducible with the code from the census.

Tool use. This work was prepared with substantial use of Claude Code (Anthropic), an agentic AI system based on large language models, working together with the author.

- *Mathematics.* The author asked which open conjectures could be checked by a computation on the author's GPU and decided which checks to run; the candidates came from a literature search carried out with Claude Code. The verification method, its calibration and the arguments in Sections 2 and 3 were worked out by Claude Code under the author's direction.
- *Writing and typesetting.* The text and the LaTeX source were written by Claude Code, in English. The note contains no figures.
- *References.* Literature was searched and sourced with Claude Code. Quotations were checked against the texts we read; texts we did not read, or read only in another version, are named as such where they are cited.
- *Code.* All computations were done by Python code written by Claude Code and run on the author's computer (AMD Ryzen 9 3900X, NVIDIA GeForce RTX 3070): determinants and Alexander polynomials by GPU code using CuPy 14.2.0; Wirtinger presentations by Regina 7.4; faces, volumes and signatures by SnapPy 3.3.2 and spherogram 2.4.1; and the verified volumes with SnapPy inside SageMath 10.9, in interval arithmetic. The census-wide runs date from 26 September 2026; the deposited scripts, at `https://doi.org/10.5281/zenodo.23156040`, are English versions of the working scripts and were run again on 4 October 2026 where stated in the deposit, with the same results. The deposit lists the exact software versions.
- *Checking.* The note was read three times independently before deposit, the third time only the sentences changed after the second reading; the findings were checked and incorporated where they held. Before deposit, our packing script checked the numbers listed in the README of the deposit against the original outputs and the fresh logs.

The author takes full responsibility for all contents.

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