THE ALEXANDER PLATEAU OF A TWO-BRIDGE KNOT IS CARRIED BY ONE COHERENT TWIST REGION

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Abstract. Let K be a two-bridge knot whose Alexander polynomial has a central plateau of half-width $m \geq 1$, i.e. $2m+1$ equal central coefficients in absolute value. We show that the reduced alternating (Conway) diagram of K has a coherently oriented twist region in which a single crossing change produces a two-bridge knot K' with $c(K') = c(K) - 2$, plateau half-width $m-1$ and $|\sigma(K')| = |\sigma(K)| - 2$, except for the trefoil, where K' is the unknot. This is a converse-type statement, for two-bridge knots, to the twist step of Azarpendar–Juhász–Kálmán, and it shows that the plateau is carried by one chain of Hartley's move T_1. The proof is an induction over Hartley's moves and uses his bottom sequences. As a by-product it gives a written-out proof of W. Chen's theorem that the Hirasawa–Murasugi bound holds for two-bridge knots and links (in the form $Z \leq |\sigma|$); in Chen's proof (arXiv and journal versions) the induction step for T_1 leaves the case of even length as "similar".

1. Introduction

For a knot K write $\Delta_K(t) = \sum_{i=0}^{2g}(-1)^i\gamma_i t^i$ (up to a unit). Fox conjectured that for alternating K the sequence (γ_i) is trapezoidal [6]; for two-bridge knots this is Hartley's theorem [9], and Murasugi proved it for alternating algebraic knots, a class that contains the two-bridge knots [15]. The *plateau half-width* $m(K)$ is the largest m with $\gamma_{g-m} = \cdots = \gamma_{g+m}$; a knot with $m(K) \geq 1$ is called a *plateau knot* below. Hirasawa and Murasugi conjectured [10, Conj. 5.11] that $m(K) \leq |\sigma(K)|/2$ for alternating K (Stoimenow had suggested the same inequality earlier and checked it for his generators up to genus 4 [21, Rem. 9.3]); W. Chen proved this for two-bridge knots [5]. Garoufalidis, Li and Yu call a knot with $m(K) \geq 1$ *properly trapezoidal*; they count 4,118 of them among the alternating knots with up to 17

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crossings and observe that all of these have nonsimplicial faces in the subdivision induced by the log coefficients of the Links–Gould polynomial [8, §2, Question 2.2]. Azarpendar, Juhász and Kálmán [1] showed that for a *twist-concentrated* link, removing a full twist from the largest coherent twist region lowers the stable length (the number of equal central coefficients, $2m+1$ for a knot) by two and the signature by two (proof of [1, Cor. 3.6]). We ask a converse-type question: does every plateau come from a coherent twist region?

Theorem A. *Let K be a two-bridge knot with $m(K) \geq 1$. Then the Conway diagram of K (the reduced alternating diagram given by the positive continued fraction, see §5) has a coherently oriented twist region in which one crossing change gives a knot K' with*

$$m(K') = m(K) - 1, \qquad |\sigma(K')| = |\sigma(K)| - 2,$$

where K' is a two-bridge knot with a reduced alternating diagram of $c(K)-2$ crossings, except for the trefoil (of either chirality), where K' is the unknot.

Here a twist region is *coherently oriented* if its two strands run through it in the same direction.

At the other extreme, Ni showed that an alternating knot of genus g whose two leading Alexander coefficients agree in absolute value is $T(2, 2g+1)$ or its mirror [17, Thm. 1.2]; there the plateau is the whole sequence, $m = g$, and Theorem A is the step $T(2, 2g+1) \to T(2, 2g-1)$.

Theorem B. *For every two-bridge knot or two-component two-bridge link L (oriented as in §2), the number Z of central steps $\alpha_i - \alpha_{i-1} = 0$ of Hartley's coefficient sequence (one less than the number of equal central coefficients) satisfies $Z \leq |\sigma(L)|$; for knots this is $m \leq |\sigma|/2$ (Chen [5]).*

Theorem B is Chen's [5, Thm. 3.5], stated there for all admissible pairs. In his proof, the induction step for T_1 ([5, Prop. 3.8, Case 2]; Proposition 3.9, Case (2), in the journal version) is written out for odd length l (from a knot to a link) and left out for even length (from a link to a knot) with "When $l = 2n$, the argument is similar and hence omitted", in both versions. Our induction does not use Chen's Theorem 3.5 and writes out both parities.

What is classical. The description of $|\gamma_i|$ as occupation numbers of the walk with steps $(-1)^{\lfloor iq/p \rfloor}$ is Hartley's [9, Thm. 1, pp. 241–243] (with half-shifted steps), in the form used here Minkus' [14, Lemma 11.1] (lemma number as quoted in [3]), and was read as a walk by Hoste [11]. Rasmussen gives the same walk formula for Δ together with σ as the difference of its end heights [18, Lemma 3.4, Props. 3.5, 3.6]. The identity (1) below is Hartley's. We use the walk only for the machine checks. Gaebler [7] proposed proving Hartley's trapezoid by induction over three transformations of (p, q) and proved special cases; the induction was left open there. He shows by continued fractions that T_1 raises the crossing number by one on pairs with $q < p$ (his valid pairs) [7, §6.3, Prop. 8(a)]; plateau width and signature are

not treated. Torisu determined all pairs of two-bridge knots and links that differ by one crossing change [23], by lens-space surgery; neither the Alexander polynomial nor the signature appears there. For unknotting number one, Kanenobu and Murakami [12] and, for rational subtangle replacements of distance at least 2, Baker and Buck [2, Thm. 5.2] place the change in the central entry x of a block $c_1, \dots, c_k, x, -c_k, \dots, -c_1$ at the end of a (not necessarily positive) continued fraction; the Alexander polynomial and the signature do not enter. When the plateau is carried by the top T_1-chain (Case 1 in §6), the bound is obtained as in Chen's proof of his Proposition 3.12, and the drop of Z and σ by two follows from his Proposition 3.2, Proposition 3.4 and Lemma 3.9 [5]; Chen does not state this descent, and the continued-fraction reading and $c(K') = c(K) - 2$ are not in his paper (for the top chain, $c(K') = c(K) - 2$ follows from Gaebler's Proposition 8(a)). What is new, as far as we could find, is the descent when the plateau is carried by a deeper T_1-chain (Case 2). This case does not reduce to Case 1 by Schubert's symmetry ($N(p/q')$ is $N(p/q)$ or its mirror image for $q' \equiv \pm q^{\pm 1} \pmod p$): for $K = N(39/14)$, with $39/14 = [2, 1, 3, 1, 2]$ (a palindrome, $14^2 \equiv 1 \bmod 39$), the coefficients are $1, 5, 9, 9, 9, 5, 1$ ($m = 1$, $|\sigma| = 2$, $c = 9$); lowering the middle entry by 2 gives $N(21/8)$ with $m = 0$, $\sigma = 0$, $c = 7$, while lowering either end entry by 2 gives $N(11/3) = N(11/4)$ with $c = 6$ and still $m = 1$, $|\sigma| = 2$. Up to mirror image it is the only plateau knot with at most nine crossings for which no admissible pair (p, q') with $q' < p$, $q' \equiv \pm q^{\pm 1} \pmod p$, is in Case 1. Also new, as far as we could find, are the reading of a T_1-chain with $k_\ell \geq 2$ as a coherently oriented twist region (Proposition 12) and, for a deeper chain, $c(K') = c(K) - 2$.

2. Hartley's framework

We follow Chen's account [5, §3] of Hartley [9]; all references to [5] are to the arXiv version (v2), except where the journal version is named. A pair (p, q) of positive integers is *admissible* if $\gcd(p, q) = 1$ and q is odd. To it belong a two-bridge knot or two-component link (a knot iff p is odd; links carry the preferred orientation of [5, §2.1], on which Δ and σ of a link depend), a length $l = l(p, q) \geq 1$, positive integers $\alpha_0, \dots, \alpha_{l-1}$ and non-negative integers $b_0, \dots, b_l$ (the *bottom sequence*). We put $\alpha_{-1} = \alpha_l = 0$ and $G_i = \alpha_i - \alpha_{i-1}$ for $0 \leq i \leq l$.

(H0) $\Delta(t) \doteq \sum_{i=0}^{l-1} (-1)^i \alpha_i t^i$ [5, Thm. 2.1], [9].

(H1) $G_i = b_i - b_{l-i}$ for $1 \leq i \leq l$ [5, Prop. 3.2]:

$$\alpha_i - \alpha_{i-1} = b_i - b_{l-i}. \tag{1}$$

(H2) The IH properties [5, Prop. 3.2]: (IH1) there are $1 \leq h \leq l$ and $r \leq h$ with $b_i = 0$ for $i > h$ and $0 \leq S_0 < S_1 < \cdots < S_r = S_{r+1} = \cdots = S_h$, where $S_{2j} = b_{h-j}$, $S_{2j+1} = b_j$; (IH2) if $h^* \geq h$ and $2j \leq h^*$ then $b_j \geq b_{h^*-j}$; (IH3) if $i < j$ and $b_i = b_j$ then b is constant on $[i, j]$. Here h in (IH2) is the h of (IH1) (an existential witness).

(H3) Every admissible pair arises from $(1,1)$ by the moves $T_1(p,q) = (p+q,q)$, $T_2(p,q) = (p,2p+q)$, $T_3(p,q) = (p,2p-q)$ (the last only for $q<p$) [5, Prop. 3.3]; T_2 keeps the knot, T_3 mirrors it [5, §3.5].

(H4) Effect on (l,b) ([5, Prop. 3.4], Hartley): T_1: $l \mapsto l+1$, b unchanged ($b_{l+1}=0$); T_2: l unchanged, $b'_i = 2\alpha_i + b_{l-i}$; T_3: l unchanged, $b'_i = 2\alpha_i - b_i$; α is unchanged by T_2, T_3 (proofs of [5, Props. 3.6, 3.7]). Since $l(1,1)=1$, and T_1 changes l by one and the parity of p while T_2, T_3 keep both, $l \equiv p \pmod 2$: l is odd for knots and even for links.

(H5) $\sigma(p,q) = \sum_{i=1}^{p-1} (-1)^{\lfloor iq/p \rfloor}$ [5, Remark in §2.3]; $\sigma(T_1 x) = \sigma(x)+1$ [5, Lemma 3.9]; $\sigma(T_2 x) = \sigma(x)$, $\sigma(T_3 x) = -\sigma(x)$ [5, §3.3–3.4]; $|\sigma(p,q)| \le l(p,q)-1$ [5, Lemma 3.13]. Since σ is a sum of $p-1$ signs, $\sigma \equiv p-1 \pmod 2$; links have odd signature.

Remark 1. The table in [5, Prop. 3.4] prints $b'_i = 2\alpha_i + b_i$ for T_2; the proof of [5, Prop. 3.6] uses $b'_i = 2\alpha_i + b_{l-i}$, which is correct and is Hartley's formula [9, (A5), p. 246]. We rebuilt Hartley's extended diagram from Chen's description and confirmed the rules of (H4) in all 1956 admissible pairs with $2 \le p < 70$, $q < 2p$ (and the printed table fails in all 1956).

For $q<p$ the pair has a canonical *word*: remove T_1 while $p>q$ (k times: $k = \lfloor p/q \rfloor$ for $q \ge 3$, $k = p-1$ for $q=1$), then T_2 while $q > 2p$ (j times), then T_3 if $q > p$ ($\varepsilon = 1$), arriving at (p'',q'') with $q'' < p''$ or $(p'',q'') = (1,1)$:

$$(p,q) = T_1^k\, T_2^j\, T_3^{\varepsilon}\, (p'',q''), \qquad (p',q) := T_2^j T_3^{\varepsilon}(p'',q'').$$

Iterating on (p'',q'') gives the full word, with one T_1-chain k_ℓ at each level ℓ. The shape $j=0, \varepsilon=0$ does not occur (it would give $q<p'$), except for $(p,1) = T_1^{p-1}(1,1)$, where $(p',q) = (p'',q'') = (1,1)$ and $b' = b'' = (1,0)$; the statements below hold for it directly (gap $= p-1$). Write $n = l(p'',q'')$, so $l(p',q) = n$ and $L := l(p,q) = n+k$; $\hat\alpha = (\alpha''_0, \dots, \alpha''_{n-1}, 0)$, $(Ry)_i = y_{n-i}$ for a sequence y on $[0,n]$, $V = \hat\alpha + R\hat\alpha$. By (H4), T_2^2 adds $2V$, so with $j = 2u$ or $2u+1$ the bottom sequence b' of (p',q) is

$$b' = \begin{cases} b'' + 2uV & (\varepsilon=0,\ j=2u,\ u\ge 1), \\ 2\hat\alpha - b'' + 2uV & (\varepsilon=1,\ j=2u), \\ 2\hat\alpha + Rb'' + 2uV & (\varepsilon=0,\ j=2u+1), \\ 2V - Rb'' + 2uV & (\varepsilon=1,\ j=2u+1), \end{cases} \tag{2}$$

and $b = b'$ (padded by k zeros) is the bottom sequence of (p,q). Hence $G_i = b'_i - b'_{L-i} =: (\Delta_k b')_i$ for $1 \le i \le L$; in the same way $(\Delta_k y)_i = y_i - y_{L-i}$ for any sequence y on $[0,n]$, padded by zeros.

3. Strict trapezoid, symmetry, and $b \le \alpha$

Lemma 2 (strict trapezoid)**.** *For every admissible pair, $G_{l-i} = -G_i$ for $1 \le i \le l-1$, α is symmetric ($\alpha_i = \alpha_{l-1-i}$), $G_i > 0$ on the left half up to the first zero, and the zeros of G form one block, symmetric about $l/2$.*

Proof. Antisymmetry is (1): $G_{l-i} = b_{l-i} - b_i = -G_i$. Telescoping, $\alpha_{l-1-i} - \alpha_i = \sum_{j=i+1}^{l-1-i} G_j = 0$. (IH2) with $h^* = l \geq h$ gives $b_i \geq b_{l-i}$ for $2i \leq l$, so $G_i \geq 0$ there ($G_0 = \alpha_0 > 0$). If $G_i = 0$ with $1 \leq i$, $2i \leq l$, then $b_i = b_{l-i}$ and (IH3) makes b constant on $[i, l-i]$, so $G_j = 0$ for $i \leq j \leq l-i$. □

Chen attributes the strict trapezoid to Hartley [5, Thm. 3.5, (3.1)] and states that the IH properties "imply the trapezoidal conjecture for two-bridge knots". Like [5, (3.1)], Lemma 2 covers links; we include its short proof from the IH properties.

Let $[e_0, e_1]$ be the central constant block of α and $Z = e_1 - e_0$ (the number of central zeros of G). Then $e_0 + e_1 = l - 1$ and $l + Z$ is odd; for a knot $Z = 2m$, for a link $Z \geq 1$ is odd.

Lemma 3 ($b \leq \alpha$)**.** *If $q < p$, then $(p, q) = T_1(p_1, q)$, an IH1 witness h of (p_1, q) satisfies $h \leq l - 1$ and is a witness for (p, q), and $b_i \leq \alpha_i$ for all i ($\alpha_l = 0 = b_l$).*

Proof. T_1 keeps b and the S-sequence uses indices $\leq h \leq l(p_1, q) = l - 1$. By (1) and $G_0 = \alpha_0 = b_0 - b_l$ (from $i = l$ and symmetry), $\alpha_i = b_i + \left[\sum_{s<i} b_s - \sum_{s \leq i} b_{l-s}\right]$. For $2(i-1) \leq l-1$, (IH2) for (p_1, q) with $h^* = l - 1$ gives $b_s \geq b_{l-1-s}$ ($2s \leq l-1$), so the bracket is ≥ 0 (recall $b_l = 0$). For $(l-1)/2 \leq i \leq l-1$, (IH2) with $s = l - 1 - i$ gives $b_i \leq b_{l-1-i} \leq \alpha_{l-1-i} = \alpha_i$. □

By (IH1), b is positive on $[0, h-1]$, zero beyond h, strictly increasing up to its maximal block $[\lceil (r-1)/2 \rceil,\ h - \lceil r/2 \rceil]$ and strictly decreasing from the end of that block to h; the neighbours of the block, where they exist, carry S_{r-1}, S_{r-2}.

Lemma 4. *In (2), $b' > 0$ on $[0, n-1]$; and $h' := \max\{i : b'_i \neq 0\}$ equals n, except in the shape $j = 0, \varepsilon = 1$ and for $(p'', q'') = (1, 1)$ with $j = 0$, where $h' = n - 1$.*

Proof. By Lemma 3 for (p'', q'') (trivial for $(1,1)$: $b'' = (1, 0)$, $\alpha'' = (1)$), $0 \leq b'' \leq \hat{\alpha}$, so the four shapes are $\geq 2V$, $\geq \hat{\alpha}$, $\geq 2\hat{\alpha}$, $\geq V$ on $[0, n-1]$, all positive there. At $i = n$: $\hat{\alpha}_n = 0$, $b''_n = 0$ (the top T_1 of (p'', q'') appended a zero), $(Rb'')_n = b''_0 > 0$, $V_n = \alpha''_{n-1} > 0$; so $b'_n = 2u\alpha''_{n-1}$, $2u\alpha''_{n-1}$, $b''_0 + \ldots$, $(2+2u)\alpha''_{n-1} - b''_0 \geq (1+2u)\alpha''_{n-1}$ respectively, which vanishes only for $u = 0$ in the second shape. For $(1,1)$ with $j = 0$, $b' = (1, 0)$. □

4. The zero count of a word

Throughout this section (p, q) is an admissible pair with $q < p$, in the notation of §2. Let Z'', $[e_0, e_1]$, $d = \alpha''_{e_0} - \alpha''_{e_0 - 1} > 0$ refer to (p'', q'') (Lemma 2), and put

$$\text{gap} = L - 2h' - 1, \qquad C = Z'' - k, \qquad \text{base} = \begin{cases} 0 & \text{knot}, \\ 1 & \text{link}. \end{cases}$$

Proposition 5 (Lemma R)**.** $Z(p,q) = \max(\text{gap}, C, \text{base})$.

Lemma 6 (lower bound)**.** $G_i = 0$ *for* $h' < i < L - h'$*, and for* $i \in I := [e_0 + 1 + k, e_1]$*. Both blocks are centred at* $L/2$*.*

Proof. The first is clear from $G_i = b'_i - b'_{L-i}$. For the second it suffices, by (2), that Δ_k of each of $\hat{\alpha}$, $R\hat{\alpha}$, V, b'', Rb'' vanishes on I. By symmetry $(R\hat{\alpha})_i = \alpha''_{i-1}$, $\alpha''_{L-i} = \alpha''_{i-k-1}$, so $\Delta_k\hat{\alpha}_i = \alpha''_i - \alpha''_{i-k-1}$ and $\Delta_k(R\hat{\alpha})_i = \alpha''_{i-1} - \alpha''_{i-k}$; for $i \in I$ all these indices lie in $[e_0, e_1]$. On $[e_0+1, e_1]$, $G'' = 0$, so $b''_i = b''_{n-i}$ by (1); as $i \mapsto n - i$ preserves this interval, $b''_{e_0+1} = b''_{e_1}$ and (IH3) makes b'' constant there. For $i \in I$ the indices i, $i-k$, $L-i = n-(i-k)$, $n-i$ all lie in $[e_0+1, e_1]$. □

Lemma 7. *Let* $q'' < p''$ *and* $x = b''_{e_0+1} - b''_{e_0}$*. If* $Z'' \geq 2$ *then* $x \in \{0, -d\}$*; if* $Z'' = 1$ *then* $-d \leq x \leq 0$*.*

Proof. By (1) at $i = e_0$ (for $e_0 = 0$ it reads $\alpha''_0 = b''_0 - b''_n$, as in the proof of Lemma 3): $b''_{e_1+1} = b''_{e_0} - d$. Take the witness $h \leq n-1$ of Lemma 3. $Z'' \geq 2$: b'' is constant ($= \beta$) on the ≥ 2 indices $[e_0+1, e_1]$, so $\beta = 0$ (zero tail) or $\beta = M$ (maximum). If $\beta = 0$ then $e_1 + 1 > h$, $b''_{e_1+1} = 0$, $b''_{e_0} = d$, $x = -d$. If $\beta = M$, then $b''_{e_1+1} = b''_{e_0} - d < M$, so the maximal block ends at e_1; if $b''_{e_0} < M$ the maximal block is exactly $[e_0+1, e_1]$, whence $h = e_0 + e_1 + 1 + (\lceil r/2 \rceil - \lceil (r-1)/2 \rceil) \geq n$, a contradiction; so $x = 0$. $Z'' = 1$ ($e_0 = n/2 - 1$, $e_1 = n/2$): $-d \leq x \leq 0$ means $b''_{e_0} \geq b''_{e_1} \geq b''_{e_1+1}$, which holds once the maximal block starts at $\mu_0 \leq e_0$; $\mu_0 = \lceil (r-1)/2 \rceil \geq n/2$ would force $r \geq n > h \geq r$. □

Lemma 8. $b'_{e_1} > b'_{e_1+1}$*; hence the maximal block of* b' *ends at or before* e_1*.*

Proof. The values of $y_{e_1} - y_{e_1+1}$ for the building blocks y are: if $Z'' = 0$ (so $e_0 = e_1$ and, by (1) at e_0, $b''_{e_0} - b''_{e_0+1} = d$): $\hat{\alpha} \mapsto d$, $R\hat{\alpha} \mapsto -d$, $V \mapsto 0$, $b'' \mapsto d$, $Rb'' \mapsto -d$, and all four shapes give d; if $Z'' \geq 1$: $\hat{\alpha} \mapsto d$, $R\hat{\alpha} \mapsto 0$, $V \mapsto d$, $b'' \mapsto x + d$, $Rb'' \mapsto x$, and the shapes give $(x+d) + 2ud$ ($u \geq 1$), $d - x + 2ud$, $2d + x + 2ud$, $(2+2u)d - x$, all positive by Lemma 7. (For $(1,1)$, $Z'' = 0$.) □

Proof of Proposition 5. By Lemma 2 the zeros form one central block, which by Lemma 6 contains a block of size $\max(\text{gap}, C)$; it remains to show $G \neq 0$ just outside. gap $\geq C$, gap ≥ 1: $G_{h'} = b'_{h'} \neq 0$. $C >$ gap, $C \geq 1$ (so $1 \leq k \leq Z''-1$): at $i^* = e_0 + k$ the building blocks give $\Delta_k\hat{\alpha} = d$, $\Delta_k R\hat{\alpha} = 0$, $\Delta_k V = d$, $\Delta_k Rb'' = x$, $\Delta_k b'' = x + d$, so $G_{i^*} \in \{x + (1+2u)d,\ (1+2u)d - x,\ (2+2u)d + x,\ (2+2u)d - x\}$, positive by Lemma 7 ($u \geq 1$ in the first shape). *Knot,* $\max(\text{gap}, C) \leq 0$: L and $n + Z''$ are odd, so $k \equiv Z'' \pmod 2$, and $C \leq 0$ gives $k \geq Z''$; so the centre $i_0 = (L-1)/2 \geq e_1$; gap ≤ 0 gives $h' \geq i_0$, so $b'_{i_0} > 0$; by Lemma 8 and (IH1) for (p', q), b' is strictly decreasing on $[i_0, h']$, and $b'_{h'} > 0 = b'_{h'+1}$, so $G_{i_0} = b'_{i_0} - b'_{i_0+1} > 0$. *Link,* $\max(\text{gap}, C) \leq 1$: the zero at $L/2$ is forced; at $i_0 = L/2 - 1$, $G_{i_0} = b'_{i_0} - b'_{i_0+2}$. Now L is even and $n + Z''$ odd, so $k \not\equiv Z'' \pmod 2$, and $C \leq 1$ gives

$k \geq Z'' - 1$. If $k \geq Z'' + 1$, then $i_0 \geq e_1$ and the same argument applies ($b'_{i_0} > b'_{i_0+1} \geq b'_{i_0+2}$). If $k = Z'' - 1$ (so $Z'' \geq 2$), $i_0 = e_1 - 1$ and the building blocks of $y_{e_1-1} - y_{e_1+1}$ take the values of Lemma 8. □

Lemma 9 (Lemma S)**.** $\mathrm{gap} + C \leq 0$. *More precisely, with* $e_0 = (n - 1 - Z'')/2 \geq 0$*:* $\mathrm{gap} + C = -2(e_0 + 1)$*, except* $-2e_0$ *in the shape* $j = 0, \varepsilon = 1$*, and* 0 *for* $(p, 1) = T_1^{p-1}(1, 1)$.

Proof. $\mathrm{gap} = n + k - 2h' - 1$ with h' from Lemma 4, and $n = Z'' + 1 + 2e_0$. □

5. Continued fractions and twist regions

A two-bridge knot K of type (p, q), $q < p$, is the closure $N(p/q)$ of the rational tangle p/q; if $p/q = [a_1; a_2, \ldots, a_s]$ with all $a_i \geq 1$ and $a_s \geq 2$, the corresponding diagram is reduced alternating with $\sum a_i$ crossings (Conway form; Schubert [20] for the normal form). We call it the *Conway diagram* of p/q. In Theorem A and its proof we take q odd, passing to the mirror image of K if necessary ($N(p/(p-q))$ is the mirror image of $N(p/q)$), and the Conway diagram of the mirror knot is the mirror image of this diagram. Each entry a_i with $i \geq 2$ gives a twist region of its own, while for $a_1 = 1$ the crossing of the first entry lies in one twist region with those of the second; e.g. $39/25 = [1; 1, 1, 3, 1, 2]$ has six entries and five twist regions. (We verified this entry by entry with spherogram for all 10922 knot fractions p/q, $q < p$, with $\sum a_i \leq 15$; `check_twist_regions.py`. The proofs below do not use it: they only use that the crossings of one entry form a chain of bigons and hence lie in one twist region.) By Kauffman, Murasugi and Thistlethwaite [13, 16, 22], $c(K) = \sum a_i$. (We checked the identification of Chen's type (p, q) with this closure against Chen's signature formula for all admissible pairs with $3 \leq p \leq 25$ odd and $q < 4p$, §8.)

Lemma 10. *If* $(p, q) = T_1^k T_2^j T_3^\varepsilon (p'', q'')$ *and* $p''/q'' = [a_1; a_2, \ldots]$*, then* $p/q = [k; 2j, a_1, a_2, \ldots]$ *if* $\varepsilon = 0$*;* $[k; 2j + 1, 1, a_1 - 1, a_2, \ldots]$ *if* $\varepsilon = 1$*,* $a_1 \geq 2$*;* $[k; 2j + 1, a_2 + 1, a_3, \ldots]$ *if* $\varepsilon = 1$*,* $a_1 = 1$*. (These expressions need not be in normal form: they may end in* 1*, and* $2j = 0$ *occurs for* $(p, 1)$*, where* $[k; 0, 1] = [k + 1]$*.)*

Proof. $q = 2jp'' + q_0$ with $q_0 = q''$ or $2p'' - q''$, $p = kq + p''$, so $p/q = k + 1/(2j + q_0/p'')$; for $\varepsilon = 1$, $q_0/p'' = 1 + (p'' - q'')/p''$ and $x/(x-1) = [1; a_1 - 1, a_2, \ldots]$ ($a_1 \geq 2$) or $= [a_2 + 1; a_3, \ldots]$ ($a_1 = 1$) for $x = p''/q''$. □

Consequently each T_1-chain with $k_\ell \geq 2$ is one entry of the continued fraction of p/q, equal to k_ℓ, lowered by 1 if the parent level has T_3 and raised by 1 at a level $(P, 1)$ ($P/1 = [k + 1]$), the other entries not depending on k_ℓ; the dependence is affine with slope 1; p depends affinely on k_ℓ with odd slope (proof of Proposition 12). (For $k_\ell = 1$ under a T_3 the entry is 0 and vanishes, third case of Lemma 10, except at the level $(2, 1)$, where it is 1.) Lowering $k_\ell \geq 3$ by 2 lowers exactly one entry by 2 (an interior zero merges its neighbours, and a last entry 1 merges into its neighbour,

$[\dots, x, 1] = [\dots, x+1]$, both without changing the sum). The *first* entry is the top level's k (or $k+1$); under a T_3 the entry $k_\ell - 1$ is never first, and never last except at a level $(P, 1)$, where it is $(k_\ell + 1) - 1 = k_\ell$. Hence if $k_\ell \geq 3$, the entry of the chain is ≥ 2, after lowering it is ≥ 1 or an interior 0, and the lowered continued fraction has non-negative entries, first and last entries ≥ 1, and sum $c(K) - 2$.

6. Proof of Theorems A and B

For an admissible pair with $q < p$ consider the statement $P(p, q)$:

(P-i) $|\sigma| \geq Z$;

(P-ii) if $Z \geq \text{base} + 2$, there is a level ℓ of the word with $k_\ell \geq 3$ (unless $(p, q) = (3, 1)$), such that the pair (p^*, q^*) obtained from the word with k_ℓ lowered by 2 has that lowered word as its own canonical word, and $Z^* = Z - 2$, $\sigma^* = \sigma - 2\operatorname{sgn}\sigma$, and its continued fraction is that of p/q with one entry lowered by 2 (after merging).

Proof of $P(p, q)$, by induction on p. By Proposition 5 and Lemma 9, $Z = \max(\text{gap}, C, \text{base})$ with $\text{gap} + C \leq 0$.

$Z = \text{base}$. (P-i) is trivial for knots; a link has odd σ, so $|\sigma| \geq 1$.

Case 1: $\text{gap} \geq \text{base} + 2$ *(then* $C \leq -\text{gap}$*).* (P-i): $\sigma = k + \sigma(p', q)$ by (H5), $|\sigma(p', q)| \leq n - 1$ (Lemma 3.13 of [5]; $l(p', q) = n$), and $\text{gap} \leq k - n + 1$ since $h' \geq n - 1$ (Lemma 4). So $\sigma \geq k - n + 1 \geq \text{gap} = Z > 0$. (P-ii): take the top level. $k \geq \text{gap} + n - 1 \geq 3$ if $n \geq 2$ or for a link ($\text{gap} \geq 3$). If $n = 1$, $k = 2$ and (p, q) is a knot, then $(p, q) = T_1^2 T_2^j (1, 1) = (4j + 3, 2j + 1)$ with $b = (j + 1, j, 0, 0)$ and $\alpha = (j + 1, 2j + 1, j + 1)$ by (H4) and (1): $Z = 0$ for $j \geq 1$, and $j = 0$ is the trefoil. With $k - 2 \geq 1$ the lowered pair $((k-2)q + p', q)$ still exceeds q, so its word is the lowered word; (p'', q''), n, e_0, the shape and h' are unchanged and L drops by 2 (so $\text{base}^* = \text{base}$): $\text{gap}^* = \text{gap} - 2$, $C^* = C + 2$, same sum ≤ 0, hence $Z^* = \text{gap}^* = Z - 2$. $\sigma^* = \sigma - 2$ with $\sigma > 0$. One entry drops by 2 (§5). Much of this case is Chen's: here more than half of the b_i vanish, the hypothesis of [5, Lemma 3.16, Prop. 3.12]; (P-i) is obtained as in Chen's proof of [5, Prop. 3.12], and $Z^* = Z - 2$, $\sigma^* = \sigma - 2$ follow from [5, Prop. 3.2, Prop. 3.4, Lemma 3.9]. Chen does not state this descent; the continued-fraction entry and $c(K') = c(K) - 2$ are not in [5]. We keep the argument above because Case 2 uses the same bookkeeping.

Case 2: $C \geq \text{base} + 2$ *(then* $\text{gap} \leq -C$*).* $Z'' = C + k \geq 3$, so $P(p'', q'')$ applies. (P-i): $\sigma = k + \varepsilon' \sigma''$ with $\varepsilon' = \pm 1$ by (H5), $|\sigma''| \geq Z''$, so $|\sigma| \geq Z'' - k = Z$. (P-ii): take the level ℓ given by $P(p'', q'')$ ($Z'' \geq 3$ excludes $(3, 1)$; $k_\ell \geq 3$ by (P-ii)). Its pair (p''^*, q''^*) has the lowered deeper word and $q''^* < p''^*$. Reassembling with the same k, j, ε: $q_0^* = q''^*$ or $2p''^* - q''^*$ lies in $(0, 2p''^*)$ and exceeds p''^* exactly when $\varepsilon = 1$, so j, ε are recovered, and $p''^* < q^*$ (as $j \geq 1$ or $\varepsilon = 1$), so k is recovered. $n^* = n - 2$, $e_0^* = e_0$, same shape, $h'^* = h' - 2$, and $L^* = L - 2$ (so $\text{base}^* = \text{base}$): $\text{gap}^* = \text{gap} + 2$, $C^* = C - 2$, same sum, so $Z^* = C^* = Z - 2$. Signature: $|\sigma''| \geq Z'' \geq k + 2$

(knot) resp. $k+3$ (link), so $\operatorname{sgn}\sigma = \varepsilon' \operatorname{sgn}\sigma''$ and $\sigma^* = \sigma - 2\operatorname{sgn}\sigma$. One entry drops by 2. □

Proof of Theorems A and B. Theorem B is (P-i); a pair with $q > p$ is reduced by T_2^{-1} (same link, same α and σ) to $q < 2p$ and, if still $q > p$, it is $T_3(p, 2p-q)$ with $2p-q < p$ (same α, $\sigma \mapsto -\sigma$); the pairs $(1,q)$ reduce to $(1,1)$, the unknot, with $Z = 0$. For Theorem A choose, by T_2 and (for the mirror) T_3, a pair with $q < p$ odd and $K = N(p/q)$ up to mirror image (mirroring a diagram preserves twist regions, their coherence, m, $|\sigma|$ and c, and the Conway diagram of the mirror knot is the mirror image, §5); let $m \geq 1$, so $Z = 2m \geq 2$. (P-ii) gives (p^*, q^*); $p^* - p$ is even (slope argument), so K' is a knot; its continued fraction has non-negative entries with sum $c(K) - 2$ and first and last entries ≥ 1 (§5), so K' is reduced alternating with $c-2$ crossings, two-bridge and hence prime (by Schubert's formula for the bridge number of a connected sum [19]), or the unknot when K is the trefoil; $Z^* = Z - 2$ gives $m - 1$, and $|\sigma^*| = |\sigma| - 2$. The lowered entry is ≥ 2 (if K is not a trefoil we have $k_\ell \geq 3$; for the trefoil it is the entry 3), so its crossings form a chain of bigons of equal sign; changing one of them and cancelling it against its neighbour by a Reidemeister II move lowers the entry by 2. So one crossing change in that twist region gives K', and the region is coherently oriented by Proposition 12. □

7. Coherent orientation

Lemma 11. *Let D_k be a knot diagram with a two-strand twist region of k crossings, the rest fixed. The region is coherently oriented iff D_{k+1} has two components.*

Proof. The four endpoints admit three pairings; inside the region the strands realise two of them, A and B, alternating with the parity of k, and the rest of the diagram induces a pairing O. If O is the third pairing, $A \cup O$ and $B \cup O$ are both one loop and the loop traverses the region in opposite directions; if $O \in \{A, B\}$, one of them is two loops and in the other the loop traverses the region twice in the same direction. □

Proposition 12. *For every T_1-chain with $k_\ell \geq 2$, the crossings of its entry lie in one twist region of the Conway diagram of a knot, and this region is coherently oriented (for an entry 1, which occurs only for $k_\ell = 2$ under a T_3: along the axis in which the entry twists).*

Proof. For $k_\ell \geq 2$ the entry of the chain is ≥ 1 (§5); its crossings form a chain of bigons and so lie in one twist region, and raising k_ℓ by one adds a crossing to that region. Mod 2, as affine functions of k_ℓ: at level ℓ, $p_\ell = k_\ell q_\ell + p'$ has slope $q_\ell \equiv 1$ and q_ℓ slope 0; one level up, $q_{\text{new}} = 2jp_{\text{old}} + q_0 \equiv q_{\text{old}}$ keeps slope 0 and $p_{\text{new}} = kq_{\text{new}} + p_{\text{old}}$ keeps slope 1. So raising k_ℓ by one flips the parity of p, i.e. turns the knot into a two-component link; Lemma 11 applies. □

8. Machine checks and data

All checks were run on the author's machine; scripts and result files are in the deposit of this paper, https://doi.org/10.5281/zenodo.23117903, and its README names the script and output file for each item. The script `checks_paper.py` (pure Python, about two minutes) recomputes every count in this list except the signature comparison in the second item (it needs SageMath with spherogram, SnapPy and Regina), the separate check in the third item, the carrying-region item and the two re-implementations in the last item (separate pure-Python scripts), the rest of the last item, and the remark.

- Hartley's extended diagram rebuilt from Chen's description (calibrated on Chen's example $(4,3)$: $\alpha=(2,2)$, $b=(2,1,0)$, $l=2$); for $p\le 160$, $q<p$: α equals the walk occupation numbers and (1), (IH3) hold (5217/5217 pairs).
- Chen's type (p,q) against the rational closure $N(p/q)$: signature formula (H5) agrees with spherogram up to one global sign (196/196 admissible pairs with $\sigma\neq 0$, $p\le 25$ odd, $q<4p$).
- Lemma 2: knots 9764/9764, links 9762/9762 ($2\le p<220$, $q<2p$); a separate check (same diagram builder) for $p<60$, $q<8p$ (5772 pairs, including 4329 with $b_l\neq 0$) found no failure.
- Lemma 3: $b\le\alpha$ in all 18201 pairs with $q<p$, $p<300$ (and fails in all 18201 with $p<q<2p$).
- Proposition 5: knots 2380/2380 ($p<160$), links 8544/8544 ($p<300$), pairs with $(p'',q'')\neq(1,1)$; Lemma 9 (exact value): 19969/19969 ($3\le p<320$, without $(p'',q'')=(1,1)$ with $j\ge 1$); Lemma 8: 17224/17224 ($p<300$, $(p'',q'')\neq(1,1)$).
- The word-level step of (P-ii) for every admissible pair of a plateau knot with $3\le q<p<400$: 6743/6743 give $Z-2$, $|\sigma|-2$ and entry sum $c-2$; raising the carrying k_ℓ by one gives a link in 6743/6743.
- Which region carries the plateau: for all two-bridge knots with $c\le 18$ and $m\ge 1$ other than the trefoil (1652, each counted once up to mirror image), a region that works in Theorem A is a largest coherently oriented twist region of the Conway diagram (1652/1652); it is a largest twist region overall in only 1435 of them (e.g. $19/4=[4;1,3]$: the region of four crossings is not coherent, the region of three carries). This is measured in the Conway diagram only, whereas [1, Def. 2.5] takes the maximum over all reduced alternating diagrams.
- Theorem A directly, for all two-bridge knots with $c\le 20$ (each chirality counted separately): every plateau knot other than the trefoil and its mirror (where K' is the unknot) has a crossing change in a twist region giving $c-2$, $m-1$ and $|\sigma|-2$ (3882 plateau knots at $c=20$); coherence is tested through the word step above, not here. Two re-implementations of the induction (`check_induction_a.py`, self-contained: 6941 pairs and the trefoil; `check_induction_b.py 170`, with the diagram builder of the

deposit: 5829 pairs with $q < p < 170$, knots and links, on which it also tests (P-i), i.e. Theorem B) found no failure.

Remark 13 (beyond two-bridge knots). In B. A. Burton's census of prime knots [4], all 20158 hyperbolic alternating prime knots with at most 19 crossings and $m = 1$ have a crossing change in a coherent twist region giving a prime alternating knot with $c-2$ crossings, $m' = 0$ and $|\sigma'| = |\sigma| - 2$ (census knots are counted up to mirror image). Measured in the census diagrams, none of them is twist-concentrated in the sense of [1, Def. 2.13] (with $\geq$; the sentence introducing [1, Cor. 3.6] writes $>$), and of the 4110 knots with $m \geq 2$, 241 are; since [1] take the largest coherent twist region over all reduced alternating diagrams and the census diagrams need not be twist-reduced, these counts are lower bounds. Whether Theorem A extends to all alternating knots is open.

Tool use. This work was prepared with substantial use of Claude Code (Anthropic), an agentic AI system based on large language models, working together with the author.

- *Mathematics.* The work was done with Claude Code under the author's direction. The proofs and the lemmas they rest on were worked out by Claude Code.
- *Writing and typesetting.* The text and the LaTeX source were written by Claude Code, in English. The paper contains no figures.
- *References.* Literature was searched and sourced with Claude Code. Statement and page numbers refer to the versions named in the bibliography; Minkus' lemma number is quoted from [3].
- *Code.* The checks of Section 8 were written and run by Claude Code in Python on the author's computer, those that need it with SageMath, Regina, SnapPy and spherogram. The scripts in the deposit, `https://doi.org/10.5281/zenodo.23117903`, are English versions of the working scripts of 26–30 September 2026, prepared by Claude Code on 17 October 2026: only names, comments and printed text were translated. They were run again on 17 October 2026, and their outputs agree with those of the working scripts. The deposit lists the software versions.
- *Checking.* Between 28 and 30 September 2026 drafts were read adversarially by separate instances of Claude Code in a series of review rounds, numbered up to 14 in our records; in the thirteenth, five separate instances read the proofs, the sources, the deposit and the changes, and the fourteenth read only the supplement in the deposit. Three of the deposited checking scripts (`check_induction_a.py`, `check_induction_b.py`, `check_lemma_T_q8p.py`) were first written by such reviewing instances. Their findings were checked against the sources or by computation and incorporated where they held. Version 2 (17 October 2026) follows an outside reading; it changes the

front matter, the script names, the citation of Chen's journal version and adds the deposit's DOI and this section; the mathematics is unchanged.

The author takes full responsibility for all contents.

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