# THE DEGREE OF THE ALEXANDER POLYNOMIAL OF TWISTED TORUS KNOTS FOR EVERY $s$

WERNER ALOIS STANGGASSINGER

ABSTRACT. For the twisted torus knot $K(p,q;r,s)$, the closure of $\delta_p^q \delta_r^{rs}$ with $0<q<p$, $\gcd(p,q)=1$, $2\le r<p$ and $s\neq 0$, we give a closed formula for the degree (highest minus lowest exponent) of the Alexander polynomial, valid for every $s$. It is expressed through the degree sequence $(d_m)$ of our earlier paper on the leading coefficient: $\deg\Delta = 2\max_m(d_m+mrs)-(q+1)(p-1)-(r-1)rs$. For $s<0$, Park and Adnan determined it in two regimes. The highest exponent comes from the earlier paper, whose two theorems used here are re-proved in an appendix; the lowest follows from the symmetry of $\Delta$, whose exact centre is fixed by the exponent sum of the braid. This note is a companion to that paper and closes the question its Remark 9 leaves open.

## 1. RESULT

Let $p>r\ge 2$, $0<q<p$, $\gcd(p,q)=1$, $s\in\mathbb{Z}\setminus\{0\}$, and let $K=K(p,q;r,s)$ be the closure of

$$\beta=\delta_p^q\,\delta_r^{rs}\in B_p,\qquad \delta_k=\sigma_1\sigma_2\cdots\sigma_{k-1}.$$

Since $\delta_r^r$ is a pure braid and $\gcd(p,q)=1$, the closure is a knot. For a Laurent polynomial $f$ write $\operatorname{top} f$ and $\operatorname{bot} f$ for its highest and lowest exponent, and $\deg f := \operatorname{top} f-\operatorname{bot} f$. This is the convention of Park and Adnan [PA25]: "the degree of the Alexander polynomial of a knot (defined as the difference between the highest power and the lowest power of the polynomial)". It is invariant under multiplication by units $\pm t^k$.

Let $(d_m)_{0\le m\le r-1}$ be the degree sequence of [St26, Theorem 6]:

$$d_m \;=\; q(p-1)-(r-1)+\max_{0\le v\le p-1}\big(v-p\;\mathrm{w}_{r-1-m}(v)\big),$$

where $\mathrm{w}_j(v):=\lfloor x_n/p\rfloor$ for the integers $x_n=v+nq$ and the step $n\ge 0$ at which $x_n$, read modulo $p$, lies in $W=\{0,1,\dots,r-1\}$ for the $(j+1)$-st

*Date*: Version 5, 4 November 2026 (Remark 4 and [Bry26] added). Version 4: 17 October 2026; version 3: 11 October 2026; versions 1 and 2: 4 October 2026 (version 2 changed only a comment in the LaTeX source).

2020 *Mathematics Subject Classification.* Primary 57K10.

This work was developed in collaboration with the AI system Claude Code (Anthropic): the choice of problem, the direction of the work and the computing hardware are due to the author (a sculptor); the formalisation, the algorithms and the proofs were developed by Claude Code (Anthropic) working with the author. The collaboration is stated as a fact, not a disclaimer; details are in the section on tool use at the end.

time (every term in $W$ counts, also consecutive ones), the term $x_0 = v$ itself counting if $v \in W$ (so $\mathrm{w}_0(v) = 0$ for $v \in W$). Thus $\mathrm{w}_j(v)$ counts how often the sequence wraps modulo $p$ up to that entry, reaching a multiple of $p$ counting as a wrap; this is the convention of [St26]. By [St26, Theorems 5, 6] (re-proved in Appendix A), $d_m + r - 1$ is the highest exponent of the polynomial $\gamma_m\,[r]_t$, where $[k]_t = 1 + t + \cdots + t^{k-1}$ and the $\gamma_m$ (rational functions of $t$ in general) are recalled in Section 3. Let $e := q(p-1) + (r-1)rs$ be the exponent sum of $\beta$.

**Theorem 1.** *For every $s \neq 0$,*

$$\deg \Delta_{K(p,q;r,s)} \;=\; 2 \max_{0 \le m \le r-1} \big(d_m + m\,r\,s\big) \;-\; (q+1)(p-1) \;-\; (r-1)\,r\,s.$$

The right-hand side is integer arithmetic in $O(pr)$ steps; no polynomial is formed.

*Remark* 2 (What was known). Park and Adnan [PA25] give a closed formula for $\Delta_{K(p,q;r,s)}$ and determine its degree for $s > 0$ [PA25, Prop. 11]; for $s < 0$ they determine it when $|rs| < q$ [PA25, Prop. 12] and in an extreme subcase of large $|rs|$ [PA25, Prop. 13], and write that for $s < 0$ describing the degree "becomes challenging". (Proposition, theorem and section numbers of [PA25] and [AP26] refer to the arXiv versions v3 and v1; in the journal version of [AP26] they are the same, and we have not read the journal version of [PA25].) Theorem 1 gives one closed formula for every $s \neq 0$. What it adds for $s < 0$ is the range covered by neither Proposition 12 nor Proposition 13: in the box $p < 31$, $-40 \le s \le -1$ that is 34 627 of 201 840 admissible cases (17.2 %; this count includes $r = 2$, see below); on the cases covered by Proposition 13 in that box the two formulas agree. For $s > 0$ and for $r|s| < q$ (Propositions 11 and 12) Section 4 compares Theorem 1 with SageMath (160 and 33 cases), not with the formulas of [PA25]. For $r = 2$ the degree is accessible in principle from Morton's closed formula [Mo06]; we make no claim of novelty for $r = 2$. Adnan and Park [AP26] give the degree for two of their non-monic families [AP26, Thms. 1.1, 1.3]; on those families Theorem 1 agrees with their degrees (Section 4).

*Remark* 3 (Relation to the companion paper). In [St26] the leading coefficient of $\Delta$ is determined; Remark 9 there explains that this fixes only the top of the Newton polygon, that the degree needs the bottom as well, and that this question remains open. Theorem 1 closes it. The bottom is not read off from a law for the valuations of the $\gamma_m$: it follows from the top through the symmetry of $\Delta$ with its exact centre (Lemma 8). This technique is not new: Kang, Matsusaka and Park determine the degree for Lorenz knots in the same way [KMP25, Cor. 6.12], and Adnan and Park use the symmetry of $\Delta$ to read the leading coefficient off the lowest term for their families [AP26, §3]. The new input is the top for every $s$, from [St26].

*Remark* 4 (Bryant's formula). After version 4 of this note, Bryant [Bry26] computed $\Delta$ by Dehn surgery for all $1 < q < p$, $1 \le r \le p-1$ and $s \in \mathbb{Z}$,

with the same braid convention as ours. His Theorem 1.4 gives the span (our deg) for every $s$:

$$\deg\Delta \;=\; 2\max_{k,i} E(k,i) - (p-1)q - (r-1)(1+rs) - (p+r-2),$$

where $E(k,i) = kq + i + rs\,N_i(k)$, with $0 \le k \le p-1$, $0 \le i \le r-1$ and $N_i(k) = \#\{1 \le m' \le k : (m'q+i) \bmod p < r\}$. For $1 < q < p$ his range contains that of Theorem 1, and as both are proved, the two formulas agree. One also sees it directly. Grouping the pairs $(k,i)$ by $m = N_i(k)$ turns the numerator of [Bry26, Thm. 1.3] into the multiset law of [St26, Theorem 5] up to a common factor $t^e$ (see the Introduction of version 6 of [St26], where $e = 0$ is checked by machine for $p \le 60$). Then the largest $kq+i$ with $N_i(k) = m$ is $d_m+r-1+e$, so $\max E = \max_m(d_m+mrs)+r-1+e$, and his formula becomes the right-hand side of Theorem 1 plus $2e$; hence $e = 0$. As integer formulas the two agree on all 371 200 cases with $p \le 30$, $1 < q < p$, $\gcd(p,q) = 1$, $2 \le r < p$ and $0 < |s| \le 40$ (`bryant/bryant_degree_check.py`; the comparison fails, as it must, when $s$ is replaced by $-s$ in his formula and when $\mathrm{w}_{r-1-m}$ is replaced by $\mathrm{w}_m$ in ours). We use nothing from [Bry26].

*Remark* 5 (Genus). As Park and Adnan use in [PA25], half the degree of the Alexander polynomial is a lower bound for the genus; they state it for $s > 0$ or $r|s| < q$, with equality for $s > 0$ [PA25, Thm. 2]. Theorem 1 extends the explicit bound $g(K) \ge \max_m(d_m + mrs) - \frac{1}{2}\big((q+1)(p-1) + (r-1)rs\big)$ to every $s$. We claim nothing about sharpness for $s < 0$.

## 2. Burau conventions

All matrices are over $\mathbb{Q}(t)$; we use $B(\sigma\tau) = B(\sigma)B(\tau)$. (With the opposite convention the statements below hold with anti-homomorphisms; in the twist paragraph of Step 3 the matrix $T$ is then the product in reverse order, which has the same $r$-th power.)

*Unreduced.* $\bar\beta(\sigma_j)$ acts on $\mathbb{Q}(t)^p$ with basis $e_0, \dots, e_{p-1}$ as the identity except for the block $\left(\begin{smallmatrix} 1-t & t \\ 1 & 0 \end{smallmatrix}\right)$ on $(e_{j-1}, e_j)$. This is the convention of Kang–Matsusaka–Park [KMP25, Def. 6.1]. Put $\omega = (1, t, \dots, t^{p-1})$ (a row) and $\mathbf{1} = (1, \dots, 1)^{\mathsf{T}}$.

*Reduced, convention S.* $S_i := S(\sigma_i)$ is the $(p-1)\times(p-1)$ identity except row $i$, which has $t$ in column $i-1$, $-t$ in column $i$ and 1 in column $i+1$ (entries outside the matrix are dropped). This is the reduced Burau matrix of Birman: [Bir74, (3-26)] with the last row and column deleted [Bir74, p. 122].

*Reduced, convention G.* $G_1$ is the identity with first column $(-t, \dots, -t)^{\mathsf{T}}$; for $i \ge 2$, $G_i$ is the identity except the block $\left(\begin{smallmatrix} 1-t & t \\ 1 & 0 \end{smallmatrix}\right)$ on rows and columns $i-1, i$ (rows and columns numbered from 1). This is the matrix returned by SageMath's method `burau_matrix` with `reduced=True`, which [St26] uses.

**Lemma 6** (Unitarity; classical, see the proof of [KMP25, Lemma 6.5])**.** *Let $\Omega = I + (1-t)L$ with $L$ the strictly lower triangular all-ones $p \times p$ matrix. Then $\bar\beta(\sigma)|_{t\to 1/t} = \Omega\,\bar\beta(\sigma^{-1})^{\mathsf T}\,\Omega^{-1}$ for all $\sigma \in B_p$.*

*Proof.* Kang, Matsusaka and Park use this identity with the same $\Omega$ and refer to Kassel–Turaev, *Braid Groups*, Theorem 3.1, which we have not consulted; the following proof is self-contained. Both sides are homomorphisms $B_p \to GL_p(\mathbb{Q}(t))$ (the right side reverses the order twice), so it suffices to check generators, i.e. $X\Omega = \Omega C$ with $X := \bar\beta(\sigma_j)(1/t)$ and $C := (\bar\beta(\sigma_j)^{-1})^{\mathsf T}$. Write $a = j-1$, $b = j$. Then $X = I + xy^{\mathsf T}$ with $x = e_b - e_a/t$, $y = e_a - e_b$, and $C = I + x'y'^{\mathsf T}$ with $x' = e_b - e_a$, $y' = e_a - e_b/t$. We need $x(\Omega^{\mathsf T}y)^{\mathsf T} = (\Omega x')y'^{\mathsf T}$. Since $(L^{\mathsf T}y)_k = \sum_{i>k} y_i$ is $-1$ at $k = a$ and 0 elsewhere, $\Omega^{\mathsf T}y = y - (1-t)e_a = t\,y'$. Since $(Lx')_i = \sum_{k<i} x'_k$ is $-1$ at $i = b$ and 0 elsewhere, $\Omega x' = x' - (1-t)e_b = t\,x$. Both sides equal $t\,xy'^{\mathsf T}$. □

**Lemma 7** (Restriction)**.** *$\omega\bar\beta(\sigma) = \omega$ and $\bar\beta(\sigma)\mathbf{1} = \mathbf{1}$ for all $\sigma$; hence $\mathbb{Q}(t)^p = \ker\omega \oplus \mathbb{Q}(t)\mathbf{1}$ with both summands invariant. Let $R(\sigma) := \bar\beta(\sigma)|_{\ker\omega}$. In the basis $v_k = te_k - e_{k+1}$ $(0 \le k \le p-2)$ of $\ker\omega$, $R(\sigma_j) = DS_jD^{-1}$ with $D = \operatorname{diag}(t^{-k})$. In particular, for every $\sigma$,*

$$\det\big(\lambda I - \bar\beta(\sigma)\big) = (\lambda - 1)\det\big(\lambda I - R(\sigma)\big),$$

$$\Lambda(\sigma) := \lim_{\lambda\to 1} \frac{\det(\lambda I - \bar\beta(\sigma))}{\lambda - 1} = \det\big(I - S(\sigma)\big).$$

*Proof.* In $\bar\beta(\sigma_j)$ column $a$ gives $t^a(1-t)+t^b = t^a$ and column $b$ gives $t^a\cdot t = t^b$, so $\omega\bar\beta = \omega$; rows $a, b$ sum to 1, so $\bar\beta\mathbf{1} = \mathbf{1}$; and $\omega\mathbf{1} = [p]_t \neq 0$. Under $\bar\beta(\sigma_j)$: $v_a \mapsto t((1-t)e_a + e_b) - te_a = -tv_a$, $v_{a-1} \mapsto te_{a-1} - (1-t)e_a - e_b = v_{a-1} + v_a$, $v_b \mapsto t^2e_a - e_{b+1} = tv_a + v_b$, and every other $v_k$ is fixed. So $R_j$ is the identity except row $a$, which is $(1, -t, t)$ in columns $(a-1, a, b)$; this is $DS_jD^{-1}$ (with 0-based row $a = j-1$ of $R$ corresponding to row $j$ of $S$). □

**Lemma 8** (Symmetry with exact centre; cf. [KMP25, Lemma 6.5], stated there for knot closures)**.** *Let $\sigma \in B_p$ have exponent sum $\epsilon$ and put $\Phi_\sigma := \Lambda(\sigma)/[p]_t$. Then*

$$\Phi_\sigma(1/t) = (-t)^{-\epsilon+p-1}\,\Phi_\sigma(t).$$

*Proof.* The proof follows [KMP25, Lemma 6.5]; we include it because the exponent is what we use. By Lemma 6, $\det(\lambda I - \bar\beta(\sigma))|_{t\to 1/t} = \det(\lambda I - \bar\beta(\sigma)^{-1}) = \det\bar\beta(\sigma)^{-1}\det(\lambda\bar\beta(\sigma) - I)$. Here $\det\bar\beta(\sigma) = (-t)^\epsilon$ and $\det(\lambda\bar\beta - I) = (-\lambda)^p\det(\lambda^{-1}I - \bar\beta)$. Divide by $\lambda - 1$ and let $\lambda \to 1$; since $(\lambda^{-1}-1)/(\lambda - 1) \to -1$, this gives $\Lambda(1/t) = (-1)^{p+1}(-t)^{-\epsilon}\Lambda(t)$. With $[p]_{1/t} = t^{-(p-1)}[p]_t$ we get $\Phi(1/t) = (-1)^{p+1}(-t)^{-\epsilon}t^{p-1}\Phi(t) = (-t)^{-\epsilon+p-1}\Phi(t)$. □

**Lemma 9** (Intertwiner $S \leftrightarrow G$)**.** *Let $n = p-1$, $f_k = e_k - e_{k+1}$ $(0 \le k \le n-1$, $e_n := 0)$, and let $P$ be the $n \times n$ matrix with rows $\rho_k = t^n e_0 + t^{n-1-k}f_k$. Then $PS_i = G_iP$ for all $i$, and $\det P = t^{n(n-1)/2}[p]_t \neq 0$. Hence $S$ and $G$ are conjugate, and every polynomial expression in the images of braids has the same determinant in both conventions.*

*Proof.* For a row vector $v$, $vS_i = v + v_c\, g$ with $c = i-1$ and $g = te_{c-1} - (t+1)e_c + e_{c+1}$. For $i = 1$: $vS_1 = v + v_0(-(t+1)e_0 + e_1)$; row 0 of $G_1 P$ is $-t\rho_0$ and row $k \geq 1$ is $\rho_k - t\rho_0$, and expanding with $(\rho_k)_0 = t^n + t^{n-1}[k=0]$ gives $PS_1 = G_1 P$. For $i \geq 2$: $g = tf_{c-1} - f_c$ and $(\rho_k)_c = [k=c]\, t^{n-1-c} - [k = c-1]\, t^{n-c}$, hence $\rho_c S_i = \rho_{c-1}$, $\rho_{c-1} S_i = t^n e_0 + (1-t)t^{n-c} f_{c-1} + t^{n-c} f_c = (1-t)\rho_{c-1} + t\rho_c$, and all other rows are unchanged; this is the action of $G_i$ on rows $c-1, c$. In the basis $(f_k)$, $P = t^n \mathbf{1}\mathbf{1}^{\mathsf{T}} + \mathrm{diag}(t^{n-1-k})$, and the matrix determinant lemma gives $\det P = t^{n(n-1)/2}[p]_t$. □

## 3. Proof of Theorem 1

Put $\Phi := \det(I - S(\beta))/[p]_t$. Theorems 5 and 6 of [St26], on which Step 3 rests, are re-proved in Appendix A; everything else is proved here or cited from [Bir74].

**Step 1 ($\Phi$ is the Alexander polynomial up to a unit).** By Burau's theorem in the form of Birman [Bir74, Thm. 3.11, (3-28)], $[p]_t\, \widetilde{\nabla}_\beta(t) = \det\left(\|\beta\|_r^\psi - \mathrm{Id}\right)$ with $\|\beta\|_r^\psi$ the reduced Burau matrix of Birman (convention $S$, Section 2), and $\widetilde{\nabla}_\beta$ the Alexander polynomial of the knot $\hat{\beta}$ [Bir74, p. 120, footnote 7]. (In the form $\Delta = \det(I - \beta_n^r(\sigma))/[n]_t$, with their reduced representation $\beta_n^r$, the same statement is [KMP25, Def. 6.2, Remark 6.3].) Since $\det(B - I) = \pm\det(I - B)$ and deg is invariant under units, $\deg \Delta_K = \mathrm{top}\,\Phi - \mathrm{bot}\,\Phi$.

**Step 2 ($\mathrm{top}\,\Phi + \mathrm{bot}\,\Phi = e - p + 1$).** By Lemma 7, $\Phi = \Lambda(\beta)/[p]_t$ exactly, so Lemma 8 gives $\Phi(1/t) = (-t)^{-e+p-1}\Phi(t)$. The highest exponent on the left is $-\mathrm{bot}\,\Phi$, on the right $\mathrm{top}\,\Phi - e + p - 1$. Hence $\mathrm{top}\,\Phi + \mathrm{bot}\,\Phi = e - p + 1$.

**Step 3 ($\mathrm{top}\,\Phi = \max_m(d_m + mrs) - (p-1)$).** Work in convention $G$; by Lemma 9, $\det(I - S(\beta)) = \det(I - G(\beta))$. As in [St26, §§2, 4] put $M = G(\delta_p^q)$, $N = G(\delta_r^r)$, $Q = (N - I)/(t^r - 1)$ and $\mathcal{P} = I - Q$, and let $\gamma_m$ be the coefficients of

$$\det\left(I - M(\mathcal{P} + uQ)\right) = \sum_{m=0}^{r-1} \gamma_m(t)\, u^m \qquad \text{(this does not involve } s\text{)};$$

this is $\det M \cdot \det(M^{-1} - \mathcal{P} - uQ)$, the determinant of [St26, ($\star$), §4].

*The twist.* The unreduced full twist on the first $r$ strands, $\bar{N} := \bar{\beta}(\delta_r^r)$, is $\bar{N} = \bar{\mathcal{P}} + t^r \bar{Q}$ with $\bar{\mathcal{P}} = \mathbf{1}_r \omega_r/[r]_t \oplus I_{p-r}$ and $\bar{Q} = (I_r - \mathbf{1}_r\omega_r/[r]_t) \oplus 0$, where $\omega_r, \mathbf{1}_r$ are the first $r$ coordinates of $\omega, \mathbf{1}$. (This is stated without proof in [St26, §4, Step 2]; here is one.) Since $\bar{\beta}(\sigma_j)$ for $j \leq r-1$ is the corresponding matrix of $B_r$ direct sum $I_{p-r}$, it suffices to treat $p = r$. By induction on $r$, using $\delta_r = \delta_{r-1}\sigma_{r-1}$, the matrix $T := \bar{\beta}(\delta_r)$ has first row $\left(1-t,\, t(1-t), \ldots, t^{r-2}(1-t),\, t^{r-1}\right)$, ones on the subdiagonal and zeros elsewhere. For such a matrix with first row $(c_0, \ldots, c_{r-1})$ the characteristic polynomial is $\lambda^r - c_0\lambda^{r-1} - \cdots - c_{r-1}$, here $\lambda^r - (1-t)\sum_{k=1}^{r-1} t^{k-1}\lambda^{r-k} - t^{r-1} = (\lambda - 1)\sum_{k=0}^{r-1} t^k \lambda^{r-1-k} = (\lambda - 1)(\lambda^r - t^r)/(\lambda - t)$. By Lemma 7 applied to $B_r$, $T\mathbf{1}_r = \mathbf{1}_r$, $\ker \omega_r$ is an invariant complement of $\mathbb{Q}(t)\mathbf{1}_r$, and the

characteristic polynomial of $T$ on $\ker\omega_r$ is $g(\lambda)=(\lambda^r-t^r)/(\lambda-t)$. Since $(\lambda-t)g(\lambda)=\lambda^r-t^r$, the Cayley–Hamilton theorem gives $T^r=t^r$ on $\ker\omega_r$. Hence $\bar N=T^r=\mathbf{1}_r\omega_r/[r]_t+t^r(I_r-\mathbf{1}_r\omega_r/[r]_t)$, the projection onto $\mathbb{Q}(t)\mathbf{1}_r$ along $\ker\omega_r$ being $\mathbf{1}_r\omega_r/[r]_t$. Since $\omega_r\mathbf{1}_r=[r]_t$, $\mathbf{1}_r\omega_r/[r]_t$ is idempotent, so $\bar{\mathcal{P}}$ and $\bar Q$ are complementary idempotents, $(\bar{\mathcal{P}}+t^r\bar Q)(\bar{\mathcal{P}}+t^{-r}\bar Q)=I$, and

$$\bar N^{\,s}=\bar{\mathcal{P}}+t^{rs}\bar Q\qquad\text{for all } s\in\mathbb{Z}.$$

$\bar Q=(\bar N-I)/(t^r-1)$ is a polynomial in $\bar N$, so all three restrict to $\ker\omega$; by Lemmas 7 and 9 the restrictions are conjugate, by one fixed matrix, to $N$, $Q$, $\mathcal{P}$. Hence $N^s=\mathcal{P}+t^{rs}Q$ and

$$\det\big(I-G(\beta)\big)=\det\big(I-MN^s\big)=\sum_m\gamma_m(t)\,t^{mrs},$$

the image of $\sum_m\gamma_m u^m$ under $u\mapsto t^{rs}$, with no unit.

*No cancellation at the top.* By [St26, Theorem 5] (Theorem 10 in Appendix A, proved there for the $\gamma_m$ as normalised above, with no sign and no unit), $\gamma_m[r]_t$ is a sum of $p$ monomials, all with coefficient $+1$. Therefore $[r]_t\det(I-G(\beta))=\sum_m\gamma_m[r]_t\,t^{mrs}$ has only positive coefficients, nothing cancels, and its highest exponent is $\max_m(d_m+r-1+mrs)$, since $d_m+r-1$ is the highest exponent of $\gamma_m[r]_t$. Since $[r]_t$ is monic of degree $r-1$, $\operatorname{top}\det(I-G(\beta))=\max_m(d_m+mrs)$. The division by $[p]_t$ (bottom 0, top $p-1$) is exact by Step 1, hence $\operatorname{top}\Phi=\max_m(d_m+mrs)-(p-1)$.

**Step 4 (assembly).**

$$\begin{aligned}\deg\Delta_K&=\operatorname{top}\Phi-\operatorname{bot}\Phi=2\operatorname{top}\Phi-(e-p+1)\\&=2\max_m(d_m+mrs)-(q+1)(p-1)-(r-1)rs.\end{aligned}\qquad\square$$

## 4. Checks

The proof above does not depend on them; they guard the conventions and the arithmetic.

- *Against SageMath's Alexander polynomial* (Seifert-matrix computation, independent of the Burau route), $3\le p\le 8$, $s\in[-6,-1]\cup\{1,2\}$, all admissible $q,r$: 640/640 in agreement (447 with $s<0$, $r|s|\ge q$; 33 with $s<0$, $r|s|<q$; 160 with $s>0$). Dropping the term $-(r-1)rs$ fails in all 640 cases.
- *Against knot Floer homology*: among the 32 009 parameter quadruples $(p,q;r,s)$ with $3\le p\le 14$, $2\le q<p$, $\gcd(p,q)=1$, $2\le r<p$, $s<0$ and at most 800 crossings in the standard braid diagram, for the 27 527 that our run evaluated ($|s|$ up to 398), the degree of the graded Euler characteristic of $\widehat{HFK}$, computed with Z. Szabó's program through SnapPy, equals Theorem 1 in every case. The other 4 482 were not evaluated: the program rejects 3 037 diagrams ("Girth number exceeds 20"), the worker process ended without a result on 1 320, and the time limit of two hours was hit on 125 (each case ran under a 3 GB memory cap, with up to 24 cases in parallel; one of the ended cases, re-run alone, finished and

agreed). Their braid diagrams have 89 to 800 crossings (median 493; for the evaluated cases 421), so the unevaluated cases lean towards larger diagrams. The run was restricted to $q \geq 2$; the SageMath comparison above contains 168 cases with $q = 1$.

- *Against the families of* [AP26]: Thm. 1.1, $K(r|s|, r|s|-1; r, s)$ with $r, |s| = 2, \ldots, 14$ (169/169, up to $p = 196$), and Thm. 1.3, $K(6n-1, 2n; 3n, -1)$ with $n = 1, \ldots, 29$ (29/29, up to $p = 173$); both theorems state the degree.
- *Step 3 exactly* in $\mathbb{Q}(t)$, $p \leq 7$, all $q, r$, $s \in [-5, 3] \setminus \{0\}$: $N^s = \mathcal{P} + t^{rs}Q$, $\det(I - G(\beta)) = \sum_m \gamma_m t^{mrs}$ and the top exponent, 0 failures in 448 cases each; a non-idempotent countercase fails.
- *Lemmas*: Lemma 6 on every generator for $p \leq 20$, Lemma 7 for $p \leq 15$, the intertwining relation of Lemma 9 in closed form for $p \leq 30$, its determinant for $p \leq 16$, and against SageMath's matrices for $p \leq 8$; deliberately wrong variants ($\Omega$ with $1 + t$, $D = \mathrm{diag}(t^k)$) fail.
- *Appendix A*: every displayed identity of A1–A7, with the notation as printed, at random complex points for all 284 admissible triples $(p, q, r)$ with $p \leq 12$: 0 failures; flipping the sign of $E$ in $\xi_3$ fails in all 284. Theorem 10 exactly in $\mathbb{Q}(t)$ for $p \leq 6$, all $q, r$, against $\det(I - M(\mathcal{P} + uQ))$.

The scripts and their outputs are deposited with this note.

## Appendix A. Proof of Theorems 5 and 6 of [St26]

In [St26, §4] the proof of Theorem 5 is given as an outline. Since Step 3 rests on it, we write it out here in full. We keep the conventions of Section 2 and of Step 3: $\bar{\beta}$ is the unreduced Burau representation, $A := \bar{\beta}(\delta_p)$, $M = G(\delta_p^q)$, $N = G(\delta_r^r)$, $Q = (N - I)/(t^r - 1)$, $\mathcal{P} = I - Q$, and $\sum_m \gamma_m u^m = \det(I - M(\mathcal{P} + uQ))$. Matrices act on columns; $\omega$ and the other row vectors act from the left. Let $J_W$ be the diagonal projection onto the coordinates $W = \{0, \ldots, r-1\}$, $\omega_W := \omega J_W$ (this is $\omega_r$ of Step 3 padded by zeros to length $p$), $\mathbf{1}_W := J_W \mathbf{1}$, $\Theta := [r]_t$, $\Pi := [p]_t$ and $\tau := 1 - t^{-q}$. We work over the field $\mathbb{Q}(t, u)$, and in A5 inside the field $\mathbb{Q}(t)(\!(y)\!)$ of Laurent series in $y := u^{-1}$, which contains $\mathbb{Q}(t, u)$.

**Theorem 10** ([St26, Theorems 5 and 6])**.** *For $0 \leq m \leq r-1$ and $j = r - 1 - m$,*

$$\gamma_m [r]_t = \sum_{v=0}^{p-1} t^{\,q(p-1)+v-p\,\mathrm{w}_j(v)}.$$

*The $p$ exponents on the right are pairwise distinct; hence $\gamma_m[r]_t$ is a sum of $p$ distinct monomials with coefficient $+1$, and its highest exponent is $d_m + r - 1$ with $d_m$ as in Section 1.*

**A1 (inverse powers).** Let $\Sigma$ be the matrix with $\Sigma e_k = e_{k-1}$ for $k \geq 1$ and $\Sigma e_0 = t^{-p} e_{p-1}$, and put $\chi := (t-1)t^{-p}$. Then $\omega\Sigma = t^{-1}\omega$, $\omega e_{p-1} = t^{p-1}$,

and for every $n \geq 1$

$$A^{-n} = \Sigma^n + \chi\, h_n\, \omega, \qquad h_n := \sum_{k=0}^{n-1} \Sigma^k e_{p-1}.$$

*Proof.* $B_i := \bar{\beta}(\sigma_i^{-1})$ sends $e_{i-1} \mapsto t^{-1}e_i$ and $e_i \mapsto e_{i-1} + (1-t^{-1})e_i$ and fixes the other $e_k$; and $A^{-1} = B_{p-1} \cdots B_1$. By induction on $k$, $C_k := B_k \cdots B_1$ satisfies $C_k e_0 = t^{-k} e_k$, $C_k e_i = e_{i-1} + (t-1)t^{i-1-k} e_k$ for $1 \leq i \leq k$, and $C_k e_i = e_i$ for $i > k$. For $k = p-1$ this says $A^{-1}e_0 = t^{1-p}e_{p-1}$ and $A^{-1}e_i = e_{i-1} + (t-1)t^{i-p}e_{p-1}$, i.e. $A^{-1} = \Sigma + \chi e_{p-1}\omega$ (column 0: $t^{-p} + \chi = t^{1-p}$); this is the claim for $n = 1$, and $\omega\chi h_1 = 1 - t^{-1}$. If $A^{-n} = \Sigma^n + \chi h_n \omega$ with $\omega\chi h_n = 1 - t^{-n}$, then $A^{-n-1} = \Sigma^{n+1} + \big(\Sigma\chi h_n + \chi(t^{-n} + \omega\chi h_n)e_{p-1}\big)\omega = \Sigma^{n+1} + \chi(\Sigma h_n + e_{p-1})\omega = \Sigma^{n+1} + \chi h_{n+1}\omega$, and $\omega\chi h_{n+1} = t^{-1}(1-t^{-n}) + (1-t^{-1}) = 1 - t^{-n-1}$. □

**A2 (the pencil).** By Step 3, $\bar{\mathcal{P}} + u\bar{\mathcal{Q}} = D_u + \nu\, \mathbf{1}_W \omega_W$ with $D_u := I + (u-1)J_W$ and $\nu := (1-u)/\Theta$. With A1 for $n = q$,

$$\mathcal{T}(u) := A^{-q} - \bar{\mathcal{P}} - u\bar{\mathcal{Q}} \;=\; \Xi + \chi\, h_q\, \omega - \nu\, \mathbf{1}_W \omega_W, \qquad \Xi := \Sigma^q - D_u.$$

Since $\omega A^{-q} = \omega$, $\omega\bar{\mathcal{P}} = \omega$, $\omega\bar{\mathcal{Q}} = 0$, $\bar{\mathcal{P}}\mathbf{1} = \mathbf{1}$ and $\bar{\mathcal{Q}}\mathbf{1} = 0$, we have $\omega\mathcal{T} = 0$ and $\mathcal{T}\mathbf{1} = 0$. So $\mathcal{T}$ maps $\mathbb{Q}(t,u)^p$ into $\ker\omega$ and kills $\mathbf{1}$; in the decomposition $\ker\omega \oplus \mathbb{Q}(t,u)\mathbf{1}$ (Lemma 7) the matrix $\mathcal{T} + z\mathbf{1}\omega$, for an indeterminate $z$, is block diagonal with blocks $\mathcal{T}|_{\ker\omega}$ and $z\Pi$. By Lemmas 7 and 9 (as in Step 3), $\mathcal{T}|_{\ker\omega}$ is conjugate to $M^{-1} - \mathcal{P} - uQ$. Hence, for $D_B := \det(M^{-1} - \mathcal{P} - uQ)$,

$$z\,\Pi\, D_B = \det\big(\mathcal{T} + z\mathbf{1}\omega\big) = \det\Xi \cdot \det\Big(I_2 + \left(\begin{smallmatrix}\omega\\ \omega_W\end{smallmatrix}\right)\Xi^{-1}\big(\chi\, h_q + z\mathbf{1},\; -\nu\mathbf{1}_W\big)\Big)$$

by the matrix determinant lemma. With $\xi_1 := \omega\Xi^{-1}\mathbf{1}$, $\xi_2 := \omega\Xi^{-1}\mathbf{1}_W$, $\xi_3 := \omega_W\Xi^{-1}\mathbf{1}$, $\xi_4 := \omega_W\Xi^{-1}\mathbf{1}_W$ the coefficient of $z$ gives

$$\Pi\, D_B = \det\Xi \cdot \big(\xi_1(1 - \nu\xi_4) + \nu\xi_2\xi_3\big).$$

$\Sigma^q$ is the matrix of the permutation $k \mapsto k - q \bmod p$, a single $p$-cycle since $\gcd(p,q) = 1$, with weight $t^{-p}$ on each step that passes from 0 to $p-1$; along the full cycle the walk descends $pq$, so the weights multiply to $t^{-pq}$. As $\Sigma^q$ has zero diagonal ($0 < q < p$), only the identity and the full cycle contribute to $\det(D_u - \Sigma^q)$, and $\det\Xi = (-1)^p(u^r - t^{-pq}) \neq 0$.

**A3 (first returns).** Put $Y := D_u^{-1} = I + (y-1)J_W$ and $\Gamma := Y\Sigma^q$. Then $\Xi = -D_u(I - \Gamma)$, so $I - \Gamma$ is invertible and $\Xi^{-1} = -\mathcal{R}\,Y$ with $\mathcal{R} := (I-\Gamma)^{-1}$. Let $\Gamma_C := (I - J_W)\Gamma = (I - J_W)\Sigma^q$; a walk of $p$ steps meets $W$, so $\Gamma_C^p = 0$ and $Z := (I - \Gamma_C)^{-1} = \sum_{k<p}\Gamma_C^k$. From $\mathcal{R}(I - \Gamma_C - J_W\Gamma) = I$ we get $\mathcal{R} = Z + \mathcal{R}J_W\Gamma Z$. Put $\mathfrak{E} := J_W\Gamma Z$ and $F := J_W\mathfrak{E}J_W$, an endomorphism of the coordinate space $\mathbb{Q}(t,u)^W$, and $F_0 := J_W\Sigma^q Z J_W$; then $F = yF_0$ with $F_0$ independent of $u$. By A5, $F_0^r = t^{-pq}I_W$, so $(I_W - F)\sum_{k<r} y^k F_0^k = (1 - y^r t^{-pq})I_W$ and $\Psi := (I_W - F)^{-1}$ exists; $F\Psi = \Psi F = \Psi - I_W$. Multiplying $\mathcal{R} = Z + \mathcal{R}J_W\Gamma Z$ by $J_W$ on the right gives $\alpha\mathcal{R}J_W = \alpha Z J_W\, \Psi$, hence for

every row $\alpha$ and column $x$

$$\alpha\mathcal{R}x = \alpha Zx + \alpha Z J_W\,\Psi\,\mathfrak{E}x.$$

Since the rows of $\Gamma_C$ lie outside $W$, $\omega_W Z = \omega_W$. Since $\omega\Sigma^q = t^{-q}\omega$ and $\omega_W\Sigma^q = y^{-1}\omega_W\Gamma$, the relation $Z = I + \Gamma_C Z$ gives $\tau\,\omega Z = \omega - y^{-1}\omega_W\mathfrak{E}$, hence $\eta := \omega Z J_W = (\omega_W - y^{-1}\omega_W F)/\tau$.

**A4 (four scalars).** Put $\kappa := \omega_W\Psi\mathbf{1}_W$, $E := \omega_W\Psi\mathfrak{E}\mathbf{1}$, $\mu := (\Theta+(y-1)\kappa)/\tau$ and $\bar y := 1-u$. Using $Y\mathbf{1}_W = y\mathbf{1}_W$, $Y\mathbf{1} = \mathbf{1} + (y-1)\mathbf{1}_W$, $\omega\mathbf{1} = \Pi$, $\omega_W\mathbf{1}_W = \Theta$, $\mathfrak{E}\mathbf{1}_W = F\mathbf{1}_W$ and A3:

$$\xi_4 = -y\,\omega_W(I_W + \Psi F)\mathbf{1}_W = -y\kappa,$$

$$\xi_2 = -y\,\eta\,\Psi\mathbf{1}_W = -\big(y\kappa - \omega_W F\Psi\mathbf{1}_W\big)/\tau = -\mu,$$

$$\xi_3 = -\big(y\Theta + E + (y-1)\omega_W\Psi F\mathbf{1}_W\big) = -\tau\mu - E,$$

$$\xi_1 = -\Big(\frac{\Pi - y^{-1}\omega_W\mathfrak{E}\mathbf{1}}{\tau} + (y-1)\,\eta\Psi\mathbf{1}_W + \eta\Psi\mathfrak{E}\mathbf{1}\Big) = -\frac{\Pi}{\tau} - \bar y\,\mu - \frac{\bar y\,E}{\tau},$$

where we used $\omega_W\Psi F\mathbf{1}_W = \kappa - \Theta$, $(y-1)\eta\Psi\mathbf{1}_W = \bar y\mu$ and $\eta\Psi\mathfrak{E}\mathbf{1} = (\bar y E + y^{-1}\omega_W\mathfrak{E}\mathbf{1})/\tau$. Since $\bar y\,y = y-1$, we have $1 - \nu\xi_4 = \tau\mu/\Theta$, and

$$\xi_1(1-\nu\xi_4) + \nu\xi_2\xi_3 = \frac{\tau\mu}{\Theta}\Big(-\frac{\Pi}{\tau} - \bar y\mu - \frac{\bar y E}{\tau}\Big) + \frac{\bar y}{\Theta}\,\mu\,(\tau\mu + E) = -\frac{\Pi\,\mu}{\Theta}.$$

With A2: $\Theta\,D_B = -\det\Xi\cdot\mu = (-1)^{p+1}(u^r - t^{-pq})\,\mu$.

**A5 (walks).** For $b\in W$ consider the walk $b$, $b-q$, $b-2q,\dots$ read modulo $p$, and let $\ell_k(b)$ be the number of steps after which it is in $W$ for the $k$-th time after the start ($\ell_0(b) = 0$). If the first return takes $\ell = \ell_1(b)$ steps and passes $\pi$ times from 0 to $p-1$, it ends at $b' := b - q\ell + p\pi \in W$, and $F_0 e_b = t^{-p\pi}e_{b'}$; so $\omega_W F_0 e_b = t^{\,b-q\ell}$, and by iteration

$$\vartheta_k := \omega_W F_0^k\mathbf{1}_W = \sum_{b\in W} t^{\,b-q\,\ell_k(b)}.$$

The walk is a single $p$-cycle meeting $W$ in $r$ points, so $\ell_{k+r}(b) = \ell_k(b) + p$, after which the walk is back at $b$ having passed $q$ times from 0 to $p-1$; hence $F_0^r = t^{-pq}I_W$ and $\vartheta_{k+r} = t^{-pq}\vartheta_k$. In $\mathbb{Q}(t)(\!(y)\!)$, $\Psi = \sum_{k\ge0} y^k F_0^k$, so $\kappa = \sum_{k\ge0} y^k\vartheta_k$ with $\vartheta_0 = \Theta$, hence $\tau\mu = \sum_{k\ge1} y^k(\vartheta_{k-1} - \vartheta_k)$ and, by the periodicity,

$$\tau\,(u^r - t^{-pq})\,\mu = u^r\sum_{k=1}^{r} y^k(\vartheta_{k-1} - \vartheta_k) = \sum_{m=0}^{r-1} u^m\,(\vartheta_j - \vartheta_{j+1}), \qquad j = r-1-m.$$

**A6 (identification with the orbit sum).** Let $O_j := \sum_{v=0}^{p-1} t^{\,v-p\,\mathrm{w}_j(v)}$. Read the orbit $v, v+q,\dots$ as integers $x_n = v + nq$, so that $\mathrm{w}_j(v) = \lfloor x_n/p\rfloor$ with $n$ the step of the $(j+1)$-st entry into $W$ (Section 1); such a step exists because the orbit is periodic and meets $W$. If this entry is at the point $b = x_n - p\lfloor x_n/p\rfloor \in W$, then $v - p\,\mathrm{w}_j(v) = b - qn$. Reversing the orbit, $v$ is reached from $b$ by $n$ steps of the downward walk of A5, and the downward walk meets $W$ exactly $j$ times in steps $1,\dots,n$; that is, $\ell_j(b) \le n < \ell_{j+1}(b)$.

Conversely, for $b \in W$ and $\ell_j(b) \le n < \ell_{j+1}(b)$, the point $v := (b-nq) \bmod p$ has its $(j+1)$-st entry into $W$ at step $n$, at $b$. So $v \mapsto (b,n)$ is a bijection between $\{0,\dots,p-1\}$ and these pairs, and

$$O_j = \sum_{b\in W} \sum_{n=\ell_j(b)}^{\ell_{j+1}(b)-1} t^{\,b-qn} = \sum_{b\in W} \frac{t^{\,b-q\ell_j(b)} - t^{\,b-q\ell_{j+1}(b)}}{1-t^{-q}} = \frac{\vartheta_j - \vartheta_{j+1}}{\tau}.$$

**A7 (assembly).** $\det M = (-t)^{q(p-1)} = (-1)^{p+1} t^{q(p-1)}$ (for $p$ even, $q$ is odd). Hence, by A4, A5 and A6,

$$\Theta \sum_m \gamma_m u^m = \Theta \det M \cdot D_B = t^{q(p-1)}(u^r - t^{-pq})\,\mu = t^{q(p-1)} \sum_{m=0}^{r-1} u^m\, O_{r-1-m},$$

which is the formula of Theorem 10. Its exponents are $\equiv q(p-1)+v \pmod p$ and therefore pairwise distinct, so nothing cancels and the highest exponent is $q(p-1) + \max_v(v - p\,\mathrm{w}_j(v)) = d_m + r - 1$. □

## Tool use

This work was prepared with substantial use of Claude Code (Anthropic), an agentic AI system based on large language models, working together with the author.

- *Mathematics.* The question is the one left open in Remark 9 of our earlier paper [St26]; it was taken up again in a survey of proof targets carried out with Claude Code in September 2026. The proof, including Appendix A, was worked out by Claude Code under the author's direction.
- *Writing and typesetting.* The text and the LaTeX source were written by Claude Code, in English. The note contains no figures.
- *References.* Literature was searched with Claude Code, including searches for prior work on the days of deposit. Quotations, definitions, theorem numbers and page references from [PA25], [AP26], [KMP25], [St26], [Bry26] and [Bir74] were checked against those texts (the arXiv versions named in the bibliography, the journal version of [AP26], the deposited file of [St26], and photographs of the pages of [Bir74]); bibliographic data were checked against Crossref, the Zenodo record of [St26], or the sources themselves.
- *Code.* The checks of Section 4 and of Remark 4 were written and run by Claude Code, in Python (with sympy and numpy), in SageMath, and for knot Floer homology with SnapPy and Z. Szabó's program as packaged for it. The deposited scripts, at `https://doi.org/10.5281/zenodo.23155383`, are English versions of the working scripts, prepared by Claude Code; the README of the deposit lists the exact software versions and states which data files were not regenerated for the deposit.

- *Checking.* Drafts of the note were read four times by separate instances of Claude Code: twice in full, once only a new proof paragraph and once only the appendix. Their findings were checked and incorporated where they held. The proofs and the numbers were then checked once more in a fresh session against the sources. The remark added in version 5 was read once more in the same way.

The author takes full responsibility for all contents.

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Wasserburg am Inn, Bavaria
*Email address*: stanggassinger@stainlesssteel4u.de